Final draft, post-refereeing, of paper published in
Educational Studies in Mathematics (2016) 92: 91-105.

DOES IT HELP TO USE MATHEMATICALLY SUPERFLUOUS BRACKETS WHEN TEACHING THE RULES FOR THE ORDER OF OPERATIONS?

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ABSTRACT

The hypothesis that mathematically superfluous brackets can be useful when teaching the rules for the order of operations is challenged. The idea of the hypothesis is that with brackets it is possible to emphasize the order priority of one operation over another. An experiment was conducted where expressions with mixed operations were studied, focusing specifically on expressions of the type $a \pm (b \times c)$ *with brackets emphasizing the multiplication compared to expressions of the type* $a \pm b \times c$ *without such brackets. Data were collected from pen and paper tests, before and after brief (about 7 min) instructions, of 169 Swedish students in year 6 and 7 (aged 12 to 13). The data do not seem to support the use of brackets to detach the middle number (b) from the first operation (* $\pm$ *) in* $a \pm b \times c$ *type of expressions.*

KEYWORDS: Mathematics education, Brackets, Order of operations

INTRODUCTION

Mathematical expressions are surrounded by conventions. One of the conventions describes the order in which operations are to be executed. As an example, in the expression $3 + 2 \times 4$ the multiplication should precede the addition, consequently this expression is equal to 11. If, on the other hand, all operations in an expression are on the same level (with respect to order priority) another convention – a left-to-right principle – is used. For instance, division and multiplication are on the same level, priority wise, and therefore none of them are to precede the other. Hence, as an example, the expression $3/2 \times 4$ is equal to 6. By the order-of-operations convention the order is the following: first brackets, second multiplication and division and third addition and subtraction. In English speaking countries the convention is typically introduced by a mnemonic like BODMAS or PEMDAS or similar[1].

Order-of-operations errors are amongst the most common arithmetic errors by students in secondary school (Blando, Kelly, Schneider & Sleeman, 1989), by students at the college level (Pappanastos, Hall & Honan, 2002) as well as by prospective teachers (Glidden, 2008). Particularly expressions of the form $a \pm b \times c$ appear to be difficult. One reason for this could be that students do not interpret mathematical expressions according to the conventions. Instead they seem to read expressions in the same way as numerals are supposed to be read, sequentially from left to right. The conflict between the sequence of natural language (left to right) and the sequence of algebra (governed by order-of-operations conventions) has been denoted by Tall and Thomas (1991) as a *parsing obstacle*.

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[1] BODMAS: (1) brackets, (2) order of (such as exponents), (3) division and multiplication, (4) addition and subtraction. PEMDAS: (1) parenthesis, (2) exponent, (3) multiplication and division, (4) addition and subtraction. Other mnemonics, like, for example, BIDMAS (brackets-index-division/multiplication-addition/subtraction) are also used. However, our study does not involve exponents.

Possibly the parsing obstacle is related to what is more visually salient (Kirshner, 1989). Landy and Goldstone (2007, 2010) have shown in a series of experiments with undergraduate students that more narrowly spaced symbols or operations tend to be solved first. Hence, as an example, in $2 \quad + \quad 3 \times 4$ the multiplication is more likely to be executed first than in the expression $2 + 3 \quad \times \quad 4$ just because of the close proximity of the operands. Indeed, symbol spacing has recently been suggested to be a useful tool to support learning of the order of operations (Gómez, Benavides-Varela, Picciano, Semenza & Dartnell, 2014).

Alternatively, instead of symbol spacing, the visual salience of the expressions could be more a question of a number bias. Liebenberg, Linchevski, Oliver and Sasman (1998) categorized arithmetic expressions of the type $a \pm b \times c$ as either to "go with the structure" (which they exemplify with $57 + 2 \times 5$) or "go against the structure" (exemplified with $13 + 7 \times 15$) or be "neutral" (exemplified with $20 + 5 \times 3$). Their assumption is that an expression that "goes with the structure" like $57 + 2 \times 5$ will result in a strong bias towards solving the operation 2×5 first (just because it is more intuitive or faster). Similarly, in the example $13 + 7 \times 15$, the bias would be towards adding 13 and 7 first, just because it is a simpler calculation than 7×15. The students' success rate relies, with their examples, on the numbers to operate with.

Linchevski and Livneh (1999) found that students had similar problems with numerical tasks, including expressions of the form $a \pm b \times c$, as with structurally similar algebraic tasks. They connected students' problems with these tasks to a lack of understanding of the structural notation – which they called a lack of *structure sense*. The concept of structure sense is related to the symbol sense described by, for example, Arcavi (1994) and Zorn (2002). However, which is subordinate the other (structure sense or symbol sense) is still under debate. Hoch (2003) and Novotná and Hoch (2008) describe structure sense as an extension of symbol sense, whereas van Stiphout, Drijvers and Gravemeijer (2013) consider structure sense to be included in symbol sense. One can claim that students need to acquire this complex feel (whatever it is called) for mathematical symbols and the structure of mathematical expressions, as well as they need to learn the conventions (including the order of operations) about how to manipulate with symbols and expressions.

The lack of structure sense, according to Linchevski and Livneh (1999), could result in students focusing on the numbers rather than on the structure or the operations in the expression. Therefore, in expressions of the type $a \pm b \times c$ the students need to detach the middle number (b) from the preceding addition/subtraction. Linchevski and Livneh (1999) also suggested a way to achieve this detachment – by inserting brackets around the multiplication, as in the expression $a \pm (b \times c)$. These brackets then show which operation should be calculated first in a chronological meaning. Thereby the convention rules for the order of operations are meant to be more explicit, and hence the structure of the expression becomes more in focus. The suggestion to use brackets in this way is exactly the hypothesis we aim to test. We have therefore designed an experiment to study the use of such brackets when students start to learn the rules for the order of operations.

Here, it is important to note that brackets can be used for different purposes in mathematical expressions. In arithmetic they can be used to define the structure of an expression, given that without the bracket the expression should be evaluated differently, for example in $3 \times (2 + 4)$. This can be described as if we for some reason want to disrupt the order of operations in $3 \times 2 + 4$, we can insert brackets (a higher priority structure element) around the addition $(2 + 4)$. In this example the brackets would break the convention (or structure) of priority of multiplication over addition. But brackets can also be used to emphasize the structure, as in $\frac{1}{(x+1)}$ or in the expression $a \pm (b \times c)$ above. In the latter cases the brackets are mathematically superfluous and removing them would not change the value of the expression. However, there may be an educational point in using brackets in that way. Marchini and Papadopulous (2011) called brackets used in this way "useless". We will call this specific (mathematically superfluous but potentially educationally meaningful) use of brackets *emphasizing brackets*.

Emphasizing brackets have indeed been found useful in mathematics teaching and learning. In particular, two studies have previously shown this – but within different contexts. Hoch and Dreyfus (2004) conclude that emphasizing brackets can help eleventh grade high school students in Israel see the structure of algebraic expressions. They let students evaluate equations with emphasizing brackets and similar equations without brackets, for example, expressions of the types $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ and $\frac{1}{4} - \frac{x}{x-1} - x = 7 + \frac{1}{4} - \frac{x}{x-1}$. They then saw that the solution frequency was higher for the equations with emphasizing brackets. Marchini and Papadopoulos (2011) showed that elementary school students could gain from using emphasizing brackets. In their experiment young students (grades 2 and 3 in Italy and Greece) compared simple algebraic expressions, and emphasizing brackets improved the number of correct answers in cases such as, for example, $\square + 4 = 9$ and $(\square + 4) = 9$. Brackets can indeed change the way in which expressions are perceived. Banerjee and Subramaniam (2005) observed students that evaluated the expression $3 \times (6 + 3 \times 5)$ with a left-to-right procedure even though the students previously had evaluated the "bare" expression $6 + 3 \times 5$ using a correct order of operations. In addition, a pair of brackets draws attention, at least when measured on adults by eye tracking (Schneider, Maruyama, Dehaene & Sigman, 2012). However, it is not clear whether the attention the brackets can draw is positive or negative for students' understanding of the convention of order priority.

In fact, there are several studies that illustrate young students' limited understanding of the use of brackets. Kieran (1979) noted students interpreting brackets being calculated first as if the brackets should appear first in a reading sense (i.e. to the left). Ayres (2000) reported on students' errors to bracket expansion exercises and suggested a connection between students' procedural errors and their working memory or cognitive load. Hewitt (2005) observed students reading out written expressions with brackets without considering the structure of the expression. In addition, Okazaki (2006) described students that struggle with necessary and unnecessary brackets.

Before we continue, let us to take a brief look at how we teach the order of operations and the use of brackets. As mentioned above mnemonics like BODMAS or PEMDAS are common in English speaking countries. However, Headlam (2013) argues, based on a study in UK, US, Japan and the Netherlands of 203 students aged between 12 to 14 years, that mnemonic rules (like BODMAS) do not help students avoid order-of-operations errors. In a Swedish context, where our study takes place, such mnemonics are rarely used to teach this convention. The convention of the order of operations is typically taught in Sweden as a set of three rules, called the priority rules; firstly brackets, secondly multiplication and division and thirdly addition and subtraction. Exponents are typically not mentioned amongst these rules. In Swedish textbooks (for grade 7) brackets are typically introduced alongside and as part of the priority rules, but the use of brackets as being either necessary or emphasizing is not discussed. We find that in two widely used Swedish teacher guides (Löwing, 2008, p. 163; McIntosh, 2008, p. 86) emphasizing brackets are suggested to be used to accentuate the order priority of multiplication over addition.

Hence, detachment of numbers and focusing on structure in expressions of the form $a \pm b \times c$ can either be triggered by emphasizing brackets (Linchevski & Livneh, 1999) or by varying symbol spacing (Landy & Goldstone, 2010; Gómez et al., 2014). However, the emphasizing brackets have an educational advantage over symbol spacing – students can easily insert such brackets by themselves without necessarily re-writing the expressions. But, on the other hand there are educational problems associated with the use of brackets (Ayres, 2000; Hewitt, 2005; Kieran, 1979; Okazaki, 2006). Both Marchini and Papadopoulos (2011) and Hoch and Dreyfus (2004) use emphasizing brackets to enhance the structure of expressions. However, in our opinion, none of these previous studies actually test the suggestion by Linchevski and Livneh (1999) on detachment. Our aim with this study is therefore to test the hypothesis that temporarily using mathematically superfluous, but potentially educationally useful, emphasizing brackets can be an appropriate way to create detachment in expressions of the type $a \pm b \times c$ when teaching the rules for the order of operations.

METHOD

This study involved 169 students aged 12 to 13 (grades 6 and 7) in nine classes in four different secondary schools in Sweden. This aim was to investigate the students' use of brackets before they had been exposed to brackets and the rules for the order of operations in school mathematics, which in the Swedish context at that time typically was in the seventh grade. A quasi-experimental design was adopted where about half of the sample (experimental group, n = 83) was exposed to emphasizing brackets and the other half was used as a control group (n = 86). Both groups were therefore exposed to a pretest, a very brief, but well controlled, instruction and a posttest during one single lesson. The intention with the short intervention was to control the experiment as well as possible and to avoid possible interference from other teaching. The entire procedure took about 30 - 40 minutes. In this way we register the students' immediate response to the instruction. Our hope was thereby to test the students' adoption of the rules for the order of operations (with and without emphasizing brackets). We note that this procedure with pretest, instruction and posttest is in contrast to the studies by Hoch and Dreyfus (2004) and Marchini and Papadoupoulos (2011) where different expressions were compared by questionnaires.

Our pre- and the posttests were of pencil-and-paper type (no calculators allowed) with a set of sixteen expressions to be evaluated, see table 1. Each test included seven expressions of the form $a \pm b \times c$ (Items No F3, F5, F6, F10, F11, F14, F16, E3, E6, E8, E10, E11, E14, E16) where we could discriminate between an order-of-operations strategy and a left-to-right strategy. In addition, and in order to be able to make comparisons with similar expressions, the tests included three expressions of the form $a \times b \pm c$ (Items F2, F8, F15, E2, E5, E15), three similar expressions, but with brackets, $a \times (b \pm c)$ (Items F4, F12, F13, E4, E12, E13) and three expressions of the type $(a \pm b) \times c$ (Items F1, F7, F9, E1, E7, E9). The results from these items have been discussed in Gunnarsson, Hernell and Sönnerhed (2012).

Table 1: The expressions to be computed in the pretest and the posttest respectively.

Pretest Item No	Expression	Posttest Item No	Expression
F1	$(3+5)\cdot 2$	E1	$(7-2)\cdot 3$
F2	$2\cdot 7+3$	E2	$2\cdot 5+4$
F3	$3+5\cdot 2$	E3	$8-3\cdot 2$
F4	$4\cdot(5-3)$	E4	$4\cdot(5-3)$
F5	$8-3\cdot 2$	E5	$2\cdot 7+3$
F6	$3+4\cdot 2$	E6	$3+4\cdot 2$
F7	$(7-1)\cdot 2$	E7	$(7-1)\cdot 2$
F8	$2\cdot 5+4$	E8	$2+5\cdot 3$
F9	$(3+7)\cdot 2$	E9	$(3+7)\cdot 2$
F10	$2+3\cdot 2$	E10	$2+3\cdot 2$
F11	$4+7\cdot 3$	E11	$4+7\cdot 3$
F12	$2\cdot(3+6)$	E12	$2\cdot(3+6)$
F13	$4\cdot(4-2)$	E13	$4\cdot(4-2)$
F14	$3+5\cdot 2$	E14	$3+5\cdot 2$
F15	$4\cdot 4-3$	E15	$4\cdot 4-3$
F16	$8-3\cdot 2$	E16	$8-3\cdot 2$

To a large extent the expressions were intentionally kept identical between pretest and posttest, but with only small variations. In alignment with the original suggestion by Linchevski and Livneh (1999), which considered expressions of the type $a \pm b \times c$, only two levels of order of operations were tested in each single expression – multiplication together with addition or subtraction. One could also anticipate that detachment could be different (possibly stronger) with division as compared to multiplication. Therefore, division was not used. All numbers were single digit and the expected answers were held positive (> 0) and reasonably low (≤ 33) in order to reduce the number of computational errors (and to be able to focus on the emphasizing brackets' effect on the detachment). The expressions in the tests were typeset with the equation editor in Word[2] in order to make symbol spacing as "neutral" as possible.

After the pretest the students were exposed to a brief teaching intervention (about 6 to 7 minutes long). The intervention, a one-way-communication-like instruction, was given by a researcher (2 classes, 42 students), or by a teacher (2 classes, 36 students) or by video clip (5 classes, 91 students). The teaching intervention given by the researcher and the teacher was written down. The same instruction and the same examples were used in the video clips. Teaching by video clip was used in the majority of the groups and was introduced in order to eliminate the risk of introducing small differences in the teaching intervention.

The instruction examples were identical and were presented in the same way in the experimental group and the control group and the presentation differed only in one essential way. The teaching intervention, for both groups, started with a claim that if there are different operations in one expression they are supposed to be conducted in the following order: first

[2] Word is a registered trademark of Microsoft Corporation

brackets, then multiplication and division and last addition and subtraction. Then four examples with necessary brackets (i.e. four examples which could have been calculated using distribution) were worked out, see Figure 1. Finally, and here the difference between the experimental group and the control group appeared, four examples were worked out which do not contain brackets, but where brackets *can* be used. The experimental group was exposed to an instruction where these examples were worked out *with* brackets, while the control group was exposed to an

The examples used in the teaching intervention

Expression	
$(5+3)\cdot 2$	expressions which require brackets
$(4+2)\cdot 4$	
$(5-1)\cdot 2$	
$3\cdot(4+3)$	
$5+3\cdot 2$	expressions which do not require brackets, but where emphasizing brackets *can* be used
$4+5\cdot 3$	
$9-4\cdot 2$	
$2\cdot 3+3$	

Figure 1. The expressions worked out as examples in the teaching intervention of how the rules for the order of operations are applied.

instruction where the examples were worked out *without* any brackets. Figure 2 illustrates in detail how one of the latter examples was worked out in the instruction for the control group and

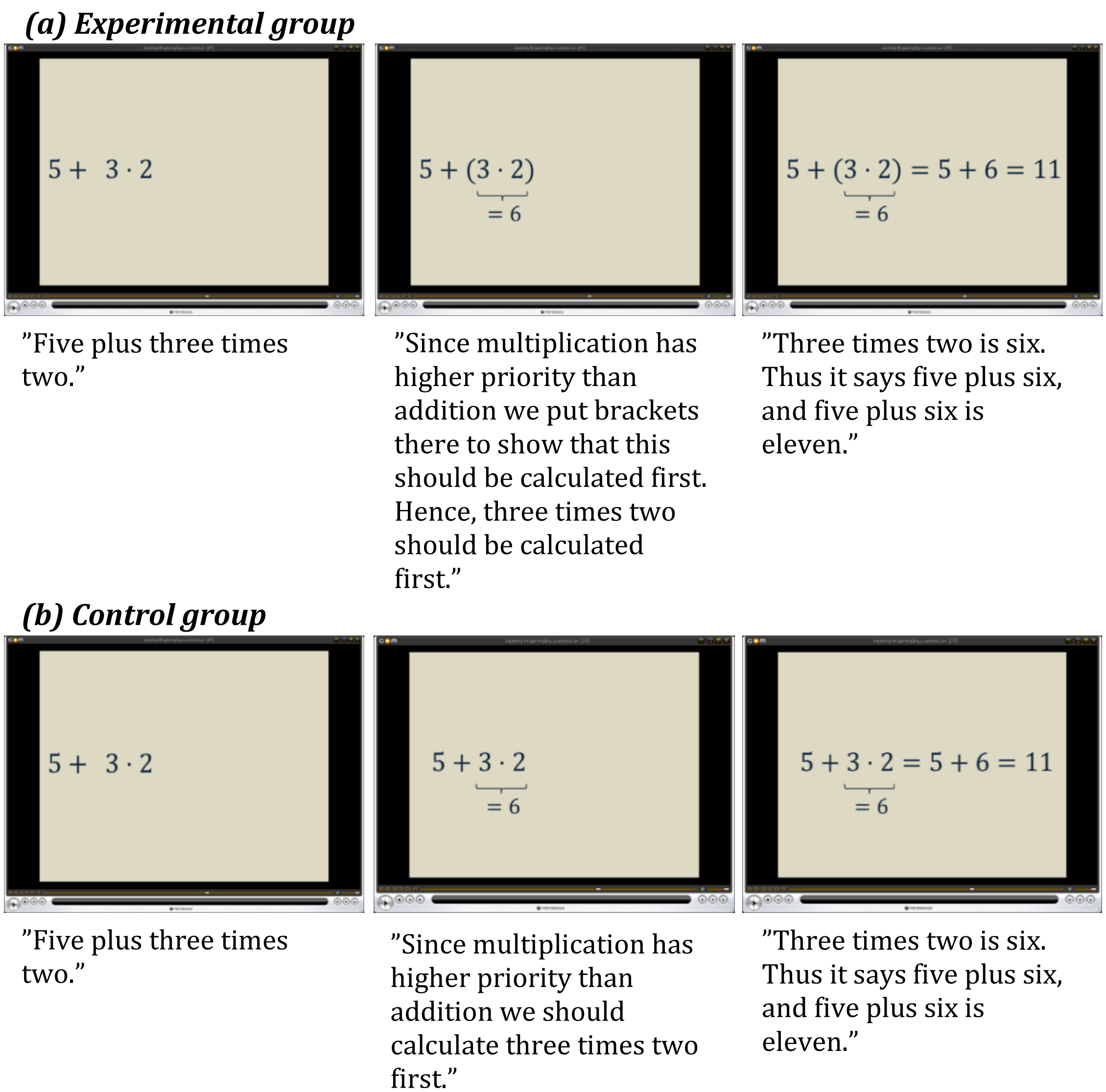

Figure 2. Screen shots from the video clips showing an example of how one of the examples that differed between the two groups was worked out in (a) the experimental group and (b) the control group, respectively. The spoken comments are written below the corresponding picture frames.

the experimental group, respectively. Note that the instruction for the first four examples did not differ between the control group and the experimental group. A posttest followed immediately after the instruction. The data from the pretest and the posttest were collected in a spread-sheet and the students' different answers were quantitatively analyzed by their type and frequency, and the hypothesis that emphasizing brackets are useful for teaching the order of operations was tested.

RESULTS

In the pretest the students scored in average 8.55 ($SD = 2.76$) right answers (out of 16) and the difference was small between the experimental group and the control group. Even though the students were assigned to either one or the other group without our prior knowledge of their abilities we do observe a small difference in the performance of the groups. Overall the average number of expressions that were correctly answered in the pretest was slightly higher in the control group ($M = 8.87$, $SD = 2.40$) than in the experimental group ($M = 8.22$, $SD = 3.04$). In the posttest the number of right answers had increased to 13.44 ($SD = 3.09$) for the control group, and to 12.12 ($SD = 2.66$) for the experimental group, respectively. Hence, from the pretest the different performance of the two groups is not statistically significant (ANOVA-pretest, F(1, 167) = 2.41, $p = .12$, $\eta^2 = .01$), but after instructions a statistically significant difference can be observed (ANOVA-posttest, F(1, 167) = 8.87, $p = .003$).

Before the instruction the video-clip-instructed groups gave on average 8.05 ($SD = 2.39$) correct answers, and the "live"-groups gave on average 9.13 ($SD = 3.04$) correct answers. After the instruction the corresponding averages were 12.44 ($SD = 2.86$) and 13.10 ($SD = 3.01$) for the video-clip-groups and the "live"-groups, respectively. ANOVA-tests of statistical significance result in F(1, 167) = 6.68, $p = .01$ (for the pretest) and F(1, 167) = 2.11, $p = .15$, $\eta^2 = .01$ (ANOVA-posttest), respectively. Even though there was a statistically significant difference before the intervention, the posttest does not show any statistically significant difference. Hence, we did not observe any significantly different results depending on which way the instruction was delivered (video-clip or "live").

If we then, regardless of which way the instruction was given, look specifically at expressions of the type $a \pm b \times c$ the two most frequent answers were either following a sequential (left-to-right) principle or the rules for the order of operations. As an example the expression $3 + 5 \times 2$, which appeared three times, twice in the pretest (F3 and F14) and once in the posttest (E14), was evaluated 504 times (169 students times 3, 3 blank answers). Out of these, 285 answers (57%) were "16" and 183 answers (36%) were "13". Hence these were by far the most common answers. As comparison, the third most frequent answer was "30" (13 answers, 3%). The rest of the answers were either blank spaces or seemed random (i.e., we could not find a systematic principle). In this analysis we take the "16"-answer as a sign of sequential left-to-right computation and the "13" answer as a sign of a computation using the order-of-operations principles. The dominance of these two kinds of answers is clear in Figure 3. In fact, overall more than 90% of the answers given in the pre- and posttests on $a \pm b \times c$ expressions were either "left-to-right" or "order-of-operations".

The tests contained in total 14 expressions of the type $a \pm b \times c$. Figure 3 shows a summary of the number of answers we could assign to a specific way of calculating these expressions. In the pretest a sequential left-to-right computation was more common, while in

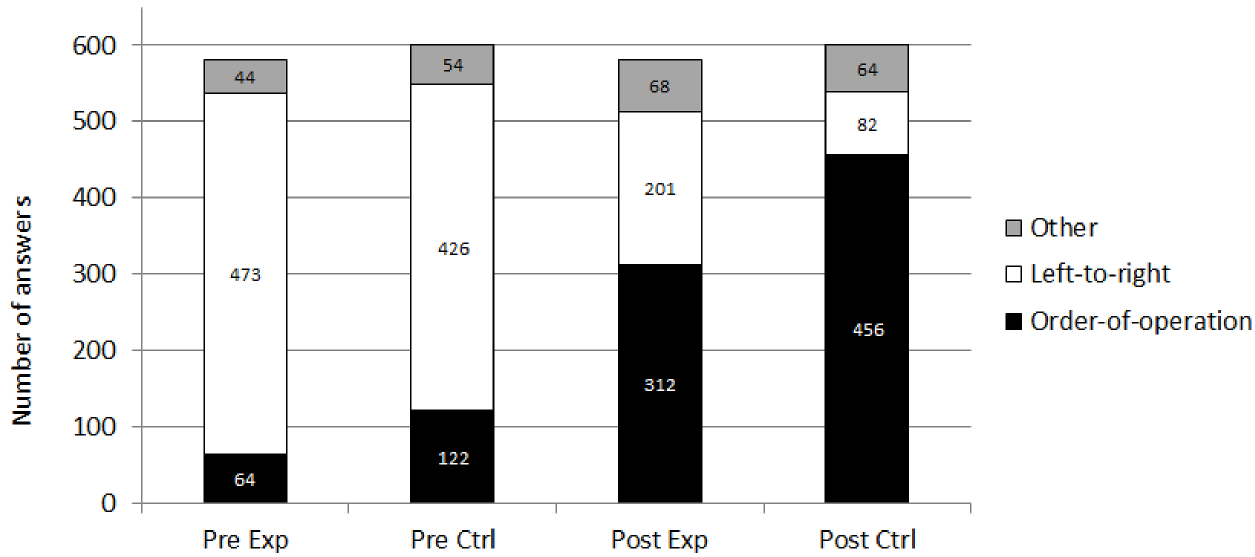

Figure 3. Summary of the number of answers to $a \pm b \times c$ type of expressions assigned to specific ways of calculating these (order-of-operations, left-to-right and other) on the pretest (Pre) and the posttest (Post) in the experimental group (Exp) and the control group (Ctrl), respectively.

the posttest a computation following the order-of-operations principles dominated. We note that both the experimental and the control group have decreased the number of answers in the left-to-right category in the posttest compared to the pretest, but also that the decrease was slightly larger in the control group. Hence, the percentage of students using a left-to-right procedure (or any other non-order-of-operations principle) in the pretest and abandoning it in favor of the rules for the order of operations is higher in the control group than in the experimental group. In both groups there has been a shift from a left-to-right processing to an order of operations procedure, but the number of students shifting appears to be larger in the control group than in the experiment group.

We use this data to test the hypothesis that temporarily using emphasizing brackets can enhance efficiency when teaching the order priority. A χ^2-test shows that with a high confidence ($\chi^2(2,\ N = 1232) = 10.20, p = .006$) we can reject the null-hypothesis that there is no difference between the two groups. The shift to a use of the order-of-operations procedure was indeed larger in the control group (without the emphasizing brackets) than in the experimental group (with the emphasizing brackets) for all identical expressions in the pre- and posttest. Hence, as will be discussed later, based on this data we do not find support for temporarily using emphasizing brackets.

We can also look only at the average number of correct answers to expressions of the form $a \pm b \times c$. By this we mean the number of answers following an appropriate order priority. We identify the average order-of-operations per group in the pretest and the posttest, respectively. There is a clear difference (as already evident from Figure 3). In the pretest the average number of answers in alignment with the order-of-operations in the experiment group is 0.77 ($SD = 1.76$) (out of seven) and in the control group 1.42 ($SD = 2.45$), respectively. This gives a difference between the groups in the pretest that borders a statistical significance (ANOVA-pretest, F(1, 167) = 3.89, $p = .05$). Whereas in the posttest the average number of answers following the order-of-operations in the experiment group is 3.76 ($SD = 2.77$) and in the control group 5.30 ($SD = 2.09$). This is definitely a significant difference (ANOVA-posttest, F(1, 167) = 16.77, $p < .001$).

Five of the expressions are identical (the same structure and the same numbers) in both the pretest and the posttest. One of these, $3 + 5 \times 2$, occurs twice in the pretest and one, $8 - 3 \times$

2, occurs twice in both the pretest and the posttest. However, the students were not always consistent in their computations, meaning that they evaluated the same expression differently at different times. In Figure 4 the percentage of students consistently shifting to an order-of-operations procedure when evaluating specific expressions is shown. Hence the figure shows for instance that 55% of the students in the experimental group that did *not* use the rules for the order of operations to evaluate $2 + 3 \times 2$ in the pretest *did* apply the appropriate order priority when evaluating the exact same expression in the posttest, and that the corresponding percentage for the same expression is 63% in the control group. The numbers in Figure 4 are based on a real change in students' answers (at least as real as we can measure), meaning that we required that a student had to answer something else than "2" (for instance "10" or leaving a blank) on *at least one* of the $8 - 3 \times 2$ expressions in the pretest *and* answer "2" to *both* of the $8 - 3 \times 2$

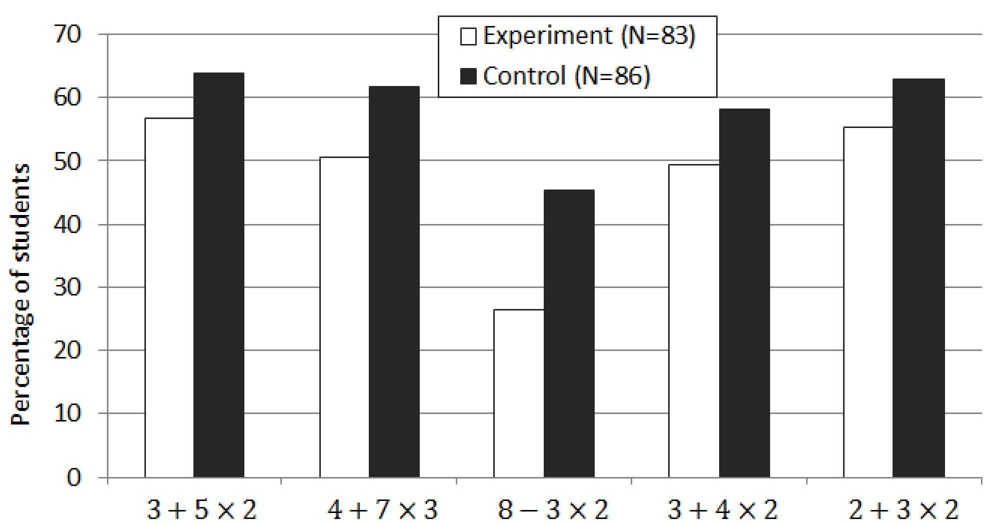

Figure 4. Percentage of students shifting from a non-order-of-operations procedure in the pretest to an order-of-operations procedure in the posttest.

expressions in the posttest in order to be included in the statistics. Hence, we count only the number of students that were *not* consistently using an order-of-operations procedure before, but were consistently using an order priority of operations after the teaching intervention. This is what we call a shift in the student's computation procedures towards using the rules for the order of operations.

As shown in Figure 5, a considerable number of the students shifted towards an order-of-operations procedure on only one, two, three or four out of five possible expressions. The different categories in Figure 5 represent the students' consistency in such a way that the categories are the number of $a \pm b \times c$ expressions in which the student has shifted towards an

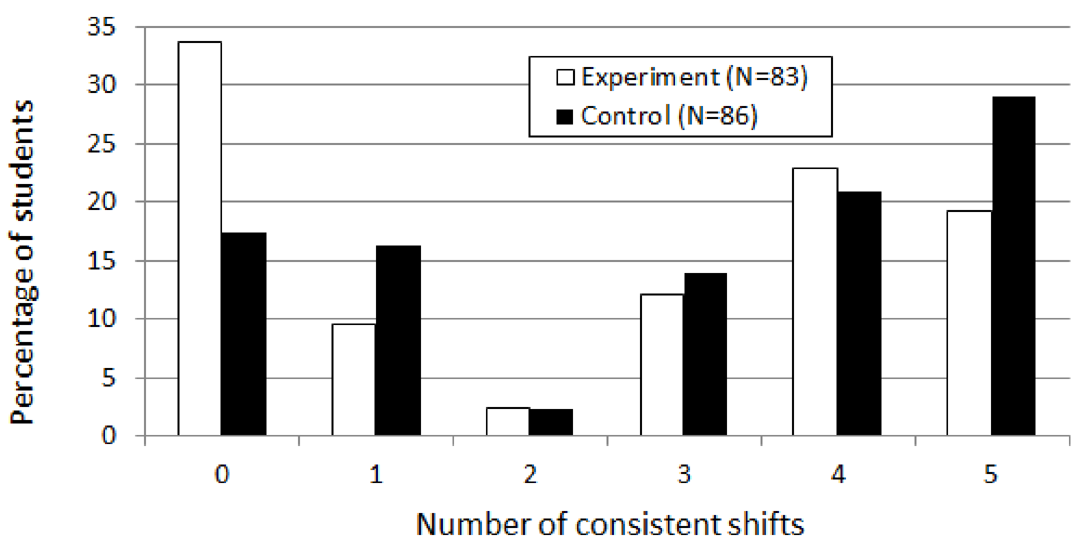

Figure 5. Percentage of students in the experimental and control groups consistently shifting to order-of-operations procedure on the identical expressions summarized in Figure 4. The categories (0 – 5) represent number of shifts (from pretest to posttest) on identical expressions.

order-of-operations procedure. There were five different reoccurring $a \pm b \times c$ expressions, i.e. expressions of this type occurring identical both in the pretest and the posttest. Thereby, a student that has calculated three out of these five reoccurring $a \pm b \times c$ expressions without using the rules for the order of operations before the instruction, and in the posttest consistently calculated these using the order-of-operations procedure, would be categorized as 3. However, the most interesting categories are the ones not shifting on any of the items (category 0) and the ones shifting towards the rules for the order of operations on all (5) identical expressions. We note that the number of students consistently shifting (category 5) is larger in the control group, and that the number of students not at all adopting the rules for the order of operations (category 0) is larger in the experimental group.

We use this data in Figure 5 to test the null hypothesis that there is no difference between the groups regarding the number of students in category 5. These are the students that can be assumed to have completely and consistently shifted from non-order-of-operations to an order-of-operations procedure. However, the number of students in category 5 was not very large. There were 16 such students in the experimental group ($n = 83$) and 25 in the control group ($n = 86$) which give $\chi^2(1, N = 41) = 1.67, p = .20$, and hence due to the small number of students the difference between the groups is not statistically significant.

Data from expressions other than $a \pm b \times c$ have been presented elsewhere (Gunnarsson et al., 2012). But we would like to take a brief look at data from the expression $a \times (b \pm c)$. We see that the second most frequent answer to this type of expression is following $a \times b \pm c$ (i.e.

without brackets). We call this effect a *bracket ignoring* effect. A summary of answers to these expressions is shown in Figure 6. As an example, expressions F13 and E13, $4 \times (4 - 2)$, typically resulted in the answer "8", but the second most frequent answer was "14" (ignoring brackets: $4 \times 4 - 2$). The answers to all three expressions of this type (F4, F12, F13 and E4, E12, E13) appear to show a decrease in bracket ignoring from pretest to posttest. In our data, see Figure 6, the decrease of bracket ignoring seems to be larger in the experimental group (16% to 2%) than in the control group (13% to 4%). However, the difference between the experimental and control groups, is not statistically significant (ANOVA-posttest, F(1, 167) = 1.23, $p = .27$, $\eta^2 = .007$).

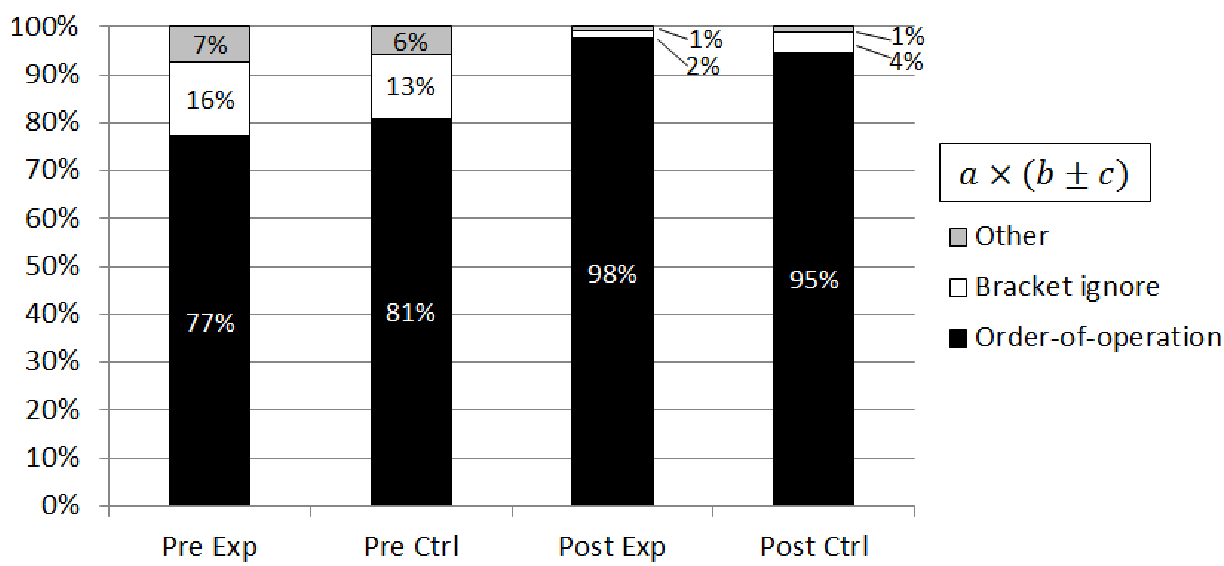

Figure 6. Distribution of student answers to $a \times (b \pm c)$ type expressions. The bracket ignoring, the second most frequent answer, substantially decreased from the pretest (Pre) to the posttest (Post) in both the experimental group (Exp) and the control group (Ctrl), respectively.

DISCUSSION

In summary, we have found that the use of mathematically superfluous brackets did not enhance the students' performance when learning the order of operations. In fact, we found the opposite. The group exposed to brackets that emphasize multiplication over addition (or subtraction) performed less well on a test where we tested the order of operations compared to a group whose instruction did not include such emphasizing brackets.

In contrast to previous studies on emphasizing brackets (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011) our study compared two teaching interventions. The previous studies compared students' success rates on *similar* expressions with and without emphasizing brackets, whereas we use the students' performance on *identical* expressions as a token of the relative efficiency of the teaching interventions. Marchini and Papadopoulos (2011) tested young pupils (grades 2-3) on arithmetic equations, and Hoch and Dreyfus (2004) tested older students' (grade 11) structure sense on algebraic equations. We instead tested students in grades 6-7, about the same as Linchevski and Livneh (1999), and their performance before and after being instructed the rules for the order of operations. Both Hoch and Dreyfus, and Marchini and Papadopoulos use emphasizing brackets to enhance the structure of the expressions. Thereby their studies essentially test the relation between emphasizing brackets and the structure sense. However, none of these previous studies actually test the suggestion by Linchevski and Livneh (1999) on detachment. We designed our study in order to test the degree to which the students applied order priority when they just had been instructed on emphasizing brackets.

In our experiment the numbers in the teaching intervention as well as in the pretest and posttest are kept low and to a large extent the computation involves single digit numbers. Hence, we expect a number biased governed detachment in the meaning of Liebenberg et al. (1998) to be less significant in our case – we consider our test tasks to be neutral in that sense. In addition, in our study we compared the students' response to the exact same expression both before and after the instruction. Hence, whether the numbers in our expressions go with or against the structure is of subordinate importance. Therefore we consider the test of the hypothesis of emphasizing brackets not to be disturbed by the potential risk of a number bias.

Instead the inconsistency, that is the number of apparent random errors, is an aggravating circumstance that obstructs a straightforward analysis of the efficiency of the teaching intervention. Students' inconsistency was noted already by Blando et al. (1989). Our way to deal with this is to analyze students' consistent shift from a sequential (left-to-right) computation to a procedure in compliance with the rules for the order of operations. The teaching intervention is considered successful if a student that used a left-to-right procedure in the pretest consistently uses an order-of-operations procedure in the posttest. However, even though we cannot claim due to these considerations of inconsistency that the shift from left-to-right to order-of-operations is statistically significant, there seems to be a difference between the experimental group and the control group, as shown both in Figure 4 and Figure 5. The restriction that we include only consistent answers (as in Figures 4 and 5) in the analysis makes the number of student answers too low to be able to establish statistical significance. If, on the other hand we include all student answers (as in Figure 3), irrespective if they were consistent on identical items, we are able to establish a statistical significant difference. Hence, we conclude that temporarily inserting emphasizing brackets in the way we did, does not seem to have supported detachment in $a \pm b \times c$ expressions. Therefore, using emphasizing brackets appears to be less efficient than *not* to use them when teaching the rules for the order of operations.

It is important to note that we do not measure if the students consciously use the convention rules for the order of operations or if there are any other motives for using that procedure. On the other hand, we do not know if the left-to-right computation procedure that dominates the students' answers before the instruction is conscious either. In particular, we have not looked at the long-term effect of the teaching intervention. We tested the students immediately after being instructed. It is therefore to some extent a remembering-and-applying-the-rule test. We note that the control group scored slightly better already from the start, as apparent from Figure 3. Hence, it is possible that this group was better at remembering the rules or perhaps more susceptible to the instruction. In addition, the non-interactive character of the intervention makes it distinctively different from an ordinary teaching situation. Since there were no opportunities for questions one could suspect more able students to more easily adopt the order-of-operations convention and thereby to perform better.

None of the examples in the instruction were presented with emphasizing brackets when first shown to the students. The emphasizing brackets were introduced, but only to the experimental group, as a part of the solution to the last four examples in Figure 1. As shown in Figure 2 this then resulted in a slightly longer instruction for the experimental group than the control group. Moreover, the experiment group was confronted to both emphasizing (mathematically superfluous) brackets and necessary brackets without an explanation of the difference between the two, whereas the control group was confronted only to necessary brackets. Hence, if we continue to speculate, a possible reason for the control group outperforming the experimental group could be that the instruction with the extra step of emphasizing brackets requires a larger cognitive load (Ayres, 2000; Sweller, 1988). Or possibly, since the students in the control group performed better in the pretest they already had a better structure sense (cf. Lüken, 2012) and more easily could apply the rules for the order of operations. Overall, one could question if the students actually got a better structure sense or if they only applied the rules more frequently.

One possible reason for the better performance of the control group could be that whereas the concept of using brackets in mathematical expressions is generally applied for showing the order of operations, when students understand the rules, the brackets can be a paradox for the students. It is possible that students first being taught the order of operations with emphasizing brackets, as in $a \pm (b \times c)$ argue that the corresponding expression without brackets $a \pm b \times c$ should be evaluated differently as the brackets are no longer there to show what should be calculated first. In the expression $a \pm (b \times c)$ the multiplication is more visually salient just because it appears narrower between symbols to the right (Kirshner, 1989), and when the salience is absent the student falls back to a (traditional) left-to-right procedure.

We could also suspect that this is related to what was observed by Banerjee and Subramaniam (2005). They noted that some of the students that had learnt to process expressions with an appropriate order of operations did not do so when the expression was put within brackets inside a somewhat more complex expression. In some way the brackets changed how the students perceived the expression. Hence, again we could interpret this as if the brackets are perceived by the students as a replacement for the rules for the order of operations, making the second (multiplication/division) and third (addition/subtraction) levels of the rules equal. We therefore suspect that students could argue that the rules are necessary in the absence of brackets, but obsolete if brackets are inserted. Similarly, if we teach the order of operations using emphasizing brackets it could be that we implicitly drill students to always see expressions with brackets. If there are no brackets in the expression from the start we put brackets there to show what should be calculated first. Hence in the end more or less all expressions will have brackets. There is then a risk that when the students later meet expressions without brackets they do not associate those expressions with the rules for the order of operations. We know that brackets draw attention (Schneider et al. 2012), but then possibly an operation without brackets does not attract the same attention.

However, we note that the students in the experimental group that were exposed to emphasizing brackets seem to have a better awareness of brackets when they appear in an expression. Bracket ignoring was observed already by Blando et al. (1989), but not found to be as frequent an error as precedence errors. We do see a decrease in both groups when it comes to the bracket ignoring effect (Figure 6), but also that the decrease in this respect appears to be (although not statistically significant) larger in the experimental group. At least the error appears to decrease (in both groups) with teaching the rules for the order of operations. Therefore it seems as if the intervention has managed to draw the students' attention to brackets, and possibly most so in the group that was exposed to both necessary and superfluous (emphasizing) brackets.

Our experiment was designed to be an extremely brief instruction and to target only the hypothesis of detachment as described by Linchevski and Livneh (1999). In doing so, we are aware that the instruction is not a complete teaching sequence on the concept of brackets or on the order of operations. Naturally this limits the chances of covering the full span of aspects of these concepts – for example, we do not contrast necessary and emphasizing brackets, nor do we discuss typical errors and how they can be avoided. However, our data indicate that even a brief instruction is sufficient to modify a behavior, or possibly the perception of expressions. We do not change the visual salience (Kirshner, 1989) of the expressions from the pre- to the posttest, but possibly the instruction changed the way the students' perceive the expression.

CONCLUSION

We have conducted an experiment in order to test the hypothesis that emphasizing brackets can be used to detach operands from operators, in our case the middle number b from the addition/subtraction in $a \pm b \times c$ expressions. However, our data do not seem to support that hypothesis, at least not in the context of immediate response to teaching the rules for the order of operations.

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