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Learning about the numerator and denominator in teacher-designed lessons

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Abstract This study concerns pupils' experience of unit and non-unit fractions of a discrete quantity during specially designed lessons. The aim was to explore pupils' understanding of operations such as *b/c of a* in lessons where the teachers were aware of some pupils' difficulties beforehand and what needed special attention. Five classes were involved in the study and 10 video-recorded lessons and written pre- and post-tests were analysed. Even though the lessons were designed for learning how to operate with both unit and non-unit fractions, we found that more pupils could solve items with unit fractions than with non-unit fractions. We found that few pupils in this study had difficulties with equal partitioning. Instead, it seemed difficult for some pupils to understand the role of the numerator and denominator and to differentiate between the *amount of parts and the amount of objects in each part*, and some pupils did not differentiate between *the numbers of units and the amount of objects within a unit*. This study identified some critical aspects that the pupils need to discern in order to learn how to operate with unit and non-unit fractions of a discrete quantity.

Keywords Fractions · Mathematics · Learning · Teaching · Variation theory

Introduction

Fractions belong to one of the most complex mathematical domains—rational numbers—that pupils encounter during pre-secondary school (Mack 1993). Pupils' understanding and learning of fractions have been widely studied (e.g., Behr et al. 1992; Kerslake 1986;

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Kieren 1993; Lamon 1996; Pitkethly and Hunting 1996; Runesson and Mok 2005; Steffe and Olive 2010). Conclusions drawn from this research are that certain aspects of fractions seem to be more difficult than others. For instance, pupils perform better in tasks with unit fractions, for example $1/3$ of 6, at an early age than on tasks with non-unit fractions, such as $2/3$ of 6 (Kieren 1993). One explanation is that the continuous model of fractions causes problems when pupils meet fractions of discrete quantities since they have difficulty identifying appropriate units (Mack 1993). Approaches to studying pupils' learning of fractions include interviews with pupils (Watanabe 1995), following particular pupils' learning trajectories during instruction (e.g., Hackenberg 2007; Olive and Vomvoridi 2006; Steffe and Olive 2010; Steffe 2004; Tzur 2004), or classroom studies of how pupils understand fractions (e.g., Ball 1993; Empson 1999). Empson (2011) argues that what children learn is sensitive to the context in which they learn, which is constituted of many factors, including most immediately the types of instructional tasks and how teachers organize students' engagement with these tasks (Empson 2011). The approach we have used in this study is to explore pupils' responses to the teacher's questions in the lesson and in written tests given before and after the lesson. Our research questions were: How do pupils experience fractions in lessons in which the teachers are aware of their difficulties and have carefully planned to overcome them? In what ways are pupils' experiences of unit and non-unit fractions exposed in the teacher-pupil interaction?

The aim of this study is thus to explore and record how pupils' experience of fractions is revealed in written tests and classroom interaction. In particular, we will note how the pupils understand and experience operations like b/c of a .

The theoretical framework used is variation theory (Marton and Booth 1997; Runesson 1999), which states that learning implies a change of understanding by discerning critical aspects of the object of learning. The theory has proven effective not only for analyses of classroom data (e.g., Häggström 2008; Kullberg 2010; Marton and Pang 2006; Runesson 1999) but also as a tool for designing lessons (Lo 2012; Pang 2010). Following this, we will describe pupils' understanding in terms of what they discern, and what is made available for them to discern.

Research on pupils' learning of unit and non-unit fractions

The present study focuses on grade 3 pupils' first formal teaching and learning of unit and non-unit fractions of a discrete quantity, b/c of a , as a part-whole construct (Behr et al. 1993). Rational numbers can be related to different sub-constructs: part-whole, measures, quotients, operators and ratios (Behr et al. 1993). It has been reported that an overreliance on the continuous part-whole model could inhibit pupils' thinking of fractions as numbers and the development of other interpretations of fractions since pupils could easily experience and treat a fraction as two whole numbers (Pitkethly and Hunting 1996). For instance, Mack (1993) recorded that pupils frequently refer to $3/4$ as "three pieces of a pizza or cake that is cut into four pieces" (p. 88). Introducing other models (measurement or sets) have shown to contribute to a rich fraction concept (Olive 2011; Steffe and Olive 2010; Tzur 2004). Petit et al. (2010) argue that the use of specific items in teaching such as $5/6$ of 6 could make pupils interpret the numerator as a whole number, seen as 5 of 6 objects.

It is clear that pupils have ideas about fractions prior to formal instruction, and that they interpret the numerator, denominator and fraction bar in various ways (Brizuela

2005). Brizuela (2005) showed that 5-6-year-olds interpreted the fraction bar itself as representing “a half”, or that a fraction (no matter the numbers on the numerator or denominator) was seen as “a half”. Pupils may think that the numbers in a fraction stand for the amount in a group, for instance that $1/3$ of 12, is “one group of three” (Empson 1999). Hence, the *denominator* is seen as indicating the amount in one of the equally partitioned groups. Similarly, pupils could experience the *numerator* in $3/4$ as “make groups of 3”, and then take all but one group (i.e. $3/4$ is one less than $4/4$) (Ball 1993).

We interpret these difficulties with fractions as a matter of not being able to differentiate between the meaning of the numerator and denominator. Another study by Olive and Vomvoridi (2006), investigating a single student’s (Tim) learning of fractions in the 6th grade, showed that at the beginning of the intervention he did not differentiate between the fractional whole ($6/6$) and the unit fraction ($1/6$). He called a circle partitioned into six parts both one-sixth and six-sixths. Furthermore, he expressed a non-unit fraction ($3/4$) as a unit fraction ($1/3$) (e.g., three one-fourths was named as one-third). They found that Tim was unable to conceptually distinguish the part from the whole, and could not establish a part-to-whole relation. Neither did he show that he understood that the parts need to be equal. When he was interviewed about a circle representing a pizza ($6/6$), where one piece ($1/6$) was cut out, Tim said that the circle showed “One-fifth, just right there, but we’re missing one.” (p. 26). Tim most likely experienced that when the number of pieces changed, the parts also changed, from being sixths at first to fifths. This is consistent with him calling the fraction $3/4$ by the unit fraction $1/3$, since there were three parts.

Partitioning, iterating and unitizing are seen as keys to developing a knowledge and understanding of fractions (Empson and Levi 2011; Lamon 1996; Wilkins and Norton 2011). There is a consensus view among researchers that the concept of unit in the domain of rational numbers changes in nature and becomes more complex (Behr et al. 1993; Lamon 1996). Ball (1993) concludes that unit fractions are easier than non-unit fractions because the compositional (making units of units) nature of fractions becomes implicit for unit fractions, however necessary to discern for non-unit fractions. She argues that when the numerator is one, it is not necessary for the pupils to make a new unit of units, but it is necessary when the numerator is greater than one. For instance, to find the answer to $3/4$ of 8 (see Fig. 1), the pupil needs to distinguish that 3 is the number of sets of the unit fraction $1/4$ of 8 and 4 is the number of groups that the whole (8) is equally partitioned into. 2 is the amount within one set ($1/4$), and 6 the amount within three sets, and hence we have a new composite unit of the whole.

When the pupils were asked to calculate $3/4$ of 8, the teachers in the present study wanted them first to determine the number of groups the whole was partitioned into, 4, and then to create a unit of three sets of $1/4$ (see Fig. 1).

Theoretical framework

Previous work framed within variation theory has demonstrated that learners’ difficulties with learning something particular could be explained by not being able to discern those aspects that are critical for learning (Marton and Pang 2006; Runesson and Mok 2005).

Fig. 1 The fraction $\frac{3}{4}$ of 8 is 3 sets of the unit fraction $\frac{1}{4}$ of 8

Variation theory (Marton and Booth 1997; Marton et al. 2004) emanates from more than 30 years of research within the phenomenographic tradition (Marton 1981; Marton and Neuman 1996). From this perspective, learning is seen as a change in one's way of *experiencing*,¹ *seeing or understanding* something (Marton and Booth 1997). How we experience something has to do with what aspects or dimensions we notice and become aware of. This discernment comes from being able to see differences and commonalities. In a similar way, Gibson and Gibson (1955) argued that perceptual learning is not a process of accumulation but of differentiation. For example, in order to understand $\frac{2}{5}$ of 10 the learner needs to differentiate the meaning of the numerator and denominator from other meanings. There are certain aspects that need to be noticed (cf., Mason 2004, 2010). A teacher can assume that it may be necessary for the learner to discern what the denominator and numerator indicate; what the whole is; that the parts the whole are partitioned into must be equal etc. However, if a pupil says during a lesson or interview that $\frac{1}{4}$ is left when $\frac{1}{5}$ is taken from a whole, it is likely that this pupil does not discern the meaning of the numerator and denominator, and does not realise that the denominator does not change even if one part is taken away, it is still fifths (Olive and Vomvoridi 2006). It is critical for the pupil to discern this aspect of fractions and differentiate it from other aspects.

Three constructs are used in variation theory to explore what is critical for students' learning from lessons. The teacher has an *intended object of learning* in mind when planning a lesson. However, this does not always coincide with what actually is made possible to learn from the lesson. The *enacted object of learning* can differ from the intended. The enacted object of learning is the researchers' perspective and analysis of what dimensions of variation and critical aspects that are made possible to experience. What students learn from the lesson and how they seem to experience the object of learning in lessons is the *lived object of learning*. In this study we analyse the enacted and lived object. We interpret the enacted and lived object of learning from the perspective of the learner. That does not mean that we fully (if possible) reveal their true understanding.

¹ Experience, seeing and understanding are used as synonyms.

Using and developing results from Learning study

Morris and Hiebert (2011) suggest the Japanese Lesson study (Lewis et al. 2006) is a system that generates knowledge about classroom learning that can be communicated, used and developed by other teachers in new contexts. The data used in the present study are derived from a larger study designed to investigate such process of transformation from one culture to another. In this case the knowledge product came from a group of teachers in Hong Kong and was used by Swedish teachers. The Hong Kong teachers had conducted a version of the Japanese Lesson study, called Learning study (Marton and Tsui 2004) aiming at enhancing their pupils' learning of fractions of discrete quantities. The Learning study was an iterative process of planning, conducting and revising a lesson plan in order to find out what was critical for their pupils' learning. It was a systematic process in which the teachers relied mainly on their own expertise and less on research literature to explore their practice. The 'results' of the Learning study were documented in terms of the critical aspects identified in the analysis of lessons and the results of pupils' tests. In addition to this, the learning goal for the lesson and some examples of tasks and test-items that had been given to the pupils in Hong Kong were provided. The critical aspects identified by the Hong Kong teachers were that the pupils needed to be aware of 1) *how many object(s) constitute(s) the whole*, and 2) *the meaning of the numerator and denominator in the fraction representing a portion of a group of objects*. The teachers described fractions of a discrete quantity as follows: "A group of objects 'a' should be divided into 'c' equal parts, and then 'b' parts are taken away $\rightarrow$ b/c of a".

The rationale behind using documentation from Hong Kong was that both the Swedish and the Hong Kong teachers were familiar with this kind of collaborative inquiry and with the underlying principles. Another, more practical, reason was that the documentation was in English and could therefore be understood by the Swedish teachers. The Swedish teachers did not, however, conduct the cyclic process, followed by the Hong Kong teachers but were asked by the researchers to take the critical aspects identified into account, as well as using examples and ideas about how to enact them in their own lessons. Whereas the overall aim of the project was to study how the documentation was used and implemented by the Swedish teachers, the specific aim and object of study of the study presented in this article is to further analyse lessons in regard to what the pupils seem to discern about unit and non-unit fractions in interaction with the teacher. It should be pointed out here that the Swedish teachers were given plenty of opportunity to adapt the documentation for use under their special conditions.

Design of the study

Five teachers from two different schools, all teaching grade three, took part in this study. Each teacher conducted two consecutive lessons (120 min for each class). This was the first formal teaching of fractions the pupils encountered. The teachers at the same school planned the two lessons together. As a resource for the planning, they had documentation from one Hong Kong study and the results of their own pupils' pre-test. The fact that the two groups of Swedish teachers had a common resource for their planning might explain why the lessons were very similar. The medium of instruction was Swedish.

The two schools were located in different parts of Sweden, in suburban areas outside one small city and one larger city. The pupils had all Swedish ethnical background. The parent(s) of the participants had given their written consent to participating in the study. The teachers were not subject specialists, but taught all subjects in primary school and had between 10 and 30 years' teaching experience. They took part on a voluntary basis and had been selected by one of the researchers since she knew that they were interested in school development. Although one of the researchers was present at some of the planning sessions, the teachers were free and encouraged to plan the lessons according to their own ideas. The researchers did not take part in the design of the lessons. The aim of the pre-tests was to find out what the pupils knew about fractions before they were formally taught so the teachers could base their lesson plan on the analysis of the test results. For the purpose of the study reported here, only classroom data (ten video-recorded lessons - two from each class) and data from the pre- and post-tests were used.

Initially we studied the results on the test in order to identify tasks that generated less learning improvement, or seemed to be most problematic. Instances in the recorded lessons that could be related to these results were then identified and coded. Some of these sequences were transcribed verbatim and analysed in detail.

Analysis of tests

The same test was given before and after the lessons. The test consisted of ten test items about unit and non-unit fractions of both continuous and discrete quantities. The items were constructed by the Hong Kong teachers themselves and may have deficiencies as regards validity and reliability. A comparison of the results from the pre- and post-tests may indicate some of the aspects that the pupils had learned from the lesson. In the present study, items from the Hong Kong study were used, $1/3$ of 9, $2/3$ of 6 and $3/5$ of 10 (see example Fig. 2).

Lisa got 6 presents on her birthday. Circle $\frac{2}{3}$ of the presents. $\frac{2}{3}$ of 6 is ____

There are 10 apples in a basket. Circle $\frac{3}{5}$ of the apples. $\frac{3}{5}$ of 10 is ____

Fig. 2 Two of the items on the pre- and post-test identical to the Hong Kong study

Lisa got 6 presents on her birthday. Circle $\frac{2}{3}$ of the presents. $\frac{2}{3}$ of 6 is ____

There are 10 apples in a basket. Circle $\frac{3}{5}$ of the apples. $\frac{3}{5}$ of 10 is ____

The particular answers to the items were categorized to provide an overview of the frequency of different answers on the pre- and post-test.

Analysis of lessons

Since the lessons were video-recorded it was possible to analyse interaction and hence what was made available for pupils to experience about fractions during the lessons. The tools used for analysis and interpretation were based on variation theory (Marton and Booth 1997). In the process of analysis the video-recorded lessons were viewed repeatedly to identify indications of the aspects of fractions which were the pupils' focii. The unit of analysis were utterances made about fractions in the interaction of the lesson. Large parts of the lessons were transcribed verbatim in Swedish and later translated into English. The analysis of pupils' answers on the test was used as a resource for analysis of the lessons, since it was assumed that the underlying reasons for the answers might be found there.

As mentioned previously, the overall aim of the larger project was to study how Swedish teachers implemented specific documentation of a lesson, the critical aspects, in their own lessons. Lessons devised from critical aspects identified by practitioners can be effective since the critical aspects takes into account both the enacted lesson (the enacted object of learning) and pupils' learning (the lived object of learning) from the lesson (cf., Empson 2011). It is the researchers' interpretation of the enacted and the lived object of learning that is reported. Since we found that the pupils demonstrated less learning in tasks with non-unit fractions, although the teachers had planned the lessons about these very carefully, we found it most valuable to analyse this more deeply. The classroom data reveal how pupils make sense of what is enacted and made possible to learn in the lessons. The data are naturalistic in character and mirror the interaction and shed light on the ways teachers encounter pupils' understandings in the classroom. In this sense, the study has ecological validity (Bryman 2004) and makes a contribution to the knowledge of the teacher-pupil interaction in regards to the content taught (cf., Empson 2011). The data production was not designed for an extensive analysis of pupils' understanding only. For instance, individual interviews with the pupils would have given better opportunities to probe questions and provided more solid grounds for such interpretations. In the classroom pupils often gave very short answers and sometimes in a low voice. The classroom data refer to a limited number of pupils and thus they cannot be seen as representing the whole group studied.

The teachers were informed by collaborative planning, their previous experience of teaching the topic, the documentation from the Hong Kong study and the frequencies of correct answers on the pre-test. It is important to note that the teachers were not informed by educational research on pupils' learning of fractions. However, the teachers had well-defined ideas when planning the lessons. For example, during the first of the two lessons equi-partitioning of a continuous whole was emphasised. The teachers partitioned a whole in different size parts to draw the pupils' attention to the fact that the parts need to be equal. The teachers varied the whole; they used different sized circles, rectangles, triangles, pizzas, apples and discrete quantities (strawberries)

and partitioned them into the same amount of parts. They also partitioned the whole in many different ways, for instance, both vertical and horizontal and in unequal parts. Another idea used from the Hong Kong study during the second lesson was to let the pupils see the same fraction ($1/4$) of a continuous whole and a discrete set simultaneously. For example, the teachers used a cake with eight strawberries on it as two different wholes. The fraction, $1/4$ was shown as part of the strawberries and of the cake to make it possible to discern differences between fractions of discrete and continuous wholes. Furthermore, inspired by the Hong Kong material, the teachers tried to direct the pupils' attention to the meaning of denominator by comparing denominators, $2/3$ and $2/4$ of a discrete set (varying the denominator) and later comparing $1/4$, $2/4$, $3/4$ and $4/4$ of a discrete set (varying the numerator) to enable them to see the difference between the numerator and denominator. The data used in this study are derived from the second lesson and focus on fractions of discrete sets.

Results

Pupils' experience of unit and non-unit fractions demonstrated in the test

The pre-test showed that operating with non-unit fractions of discrete quantities was more difficult than operating with unit fractions (see Table 1). The pre-test showed that 24 of the 59 participating pupils, that is almost half of them, were able to solve the task $1/3$ of 9 before they were formally taught. More than twice as many were able to solve this task after the lesson. But tasks with non-unit fractions seemed to have been difficult. Very few solved the two tasks, $2/3$ of 6 and $3/5$ of 10 correctly on the pre-test (only 12 and 4 respectively). Progress was made between the pre- and post-test (from 12 to 26 pupils, and from 4 to 21 pupils). Twice as many were able to solve $2/3$ of 6, and five times as many $3/5$ of 10 after the lesson. Still, less than half of them (26 and 21 pupils respectively) solved the tasks correctly after being taught.

A diversity of incorrect answers was found on the pre- and post-test on items with non-unit fractions (Tables 2 and 3). However, the variation in answers decreased between the pre- and post-test, and some answers did not appear in the post-test. On the pre-test some answers were more common than others. For example, 18 (of 59) pupils answered 3 to the task $2/3$ of 6. Other common answers were 2 and 5. Twelve pupils answered with one of these numbers. Certain incorrect answers even increased from the pre- to the post-test. For example, three more pupils gave the answer 2 in the post-test than in the pre-test. The answer 3 was still the most common incorrect answer to $2/3$ of 6. Sixteen pupils gave this answer on the post-test.

The task $3/5$ of 10 seemed to be most difficult since fewer pupils answered correctly on both the pre- and post-test compared to the other two tasks. The most

Table 1 Rate of correct answers on three items on pre- and post-tests. The figures show the number of pupils (percent in parentheses) in all five classes $N=59$

Task	Pre-test	Post-test
$1/3$ of 9	24 (40.7)	50 (84.7)
$2/3$ of 6	12 (20.3)	26 (44.1)
$3/5$ of 10	4 (6.8)	21 (35.6)

Table 2 Rate of correct and incorrect answers to the task $2/3$ of 6. The figures show the number of pupils (percent in parentheses), $N=59$

Answer	Pre-test	Post-test
4 (correct answer)	12 (20.3)	26 (44.1)
3	18 (30.5)	16 (27.1)
2	6 (10.2)	9 (15.3)
6	3 (5.1)	4 (6.8)
5	6 (10.2)	–
1	2 (3.4)	–
10	1 (1.7)	–
12	1 (1.7)	–
$4\ 2/3$	1 (1.7)	–
$4/3$	–	1 (1.7)
No answer	9 (15.3)	3 (5.1)

common answer on the pre-test to $3/5$ of 10 was 5; 13 pupils answered in this way. Other common answers were 7, 8 and 2 (8, 6, and 5 pupils respectively). The variation of answers decreased between the pre- and post-test, in the same way as with $2/3$ of 6. As many as 15 pupils gave the incorrect answer, 2, on the post test, i.e. an increase. Two or more pupils also answered 5 (7 pupils), 3 (3 pupils) and 2 (2 pupils) on the post-test.

Fewer pupils gave ‘no answer’ (on both tasks) on the post-test compared to the pre-test. Nine pupils gave no answer to the task $3/5$ of 10 after being taught, and three pupils did not give an answer to $2/3$ of 6. From the results of the pre- and post-test we conclude that the teaching of fractions during the two consecutive lessons made an impact on the pupils’ learning since the pupils answered differently, and the variation in answers on the post-test decreased compared to the pre-test. One answer in particular, 2, increased after teaching. Why? Could the increase in a particular answer be related to the lessons? In order to gain a deeper understanding of how pupils experience the numerator and

Table 3 Rate of correct and incorrect answers to the task $3/5$ of 10. The figures show the number of pupils (percent in parentheses), $N=59$

Answer	Pre test	Post test
6 (correct answer)	4 (6.8)	21 (35.6)
2	5 (8.5)	15 (25.4)
5	13 (22)	7 (11.9)
3	4 (6.8)	3 (5.1)
9	1 (1.7)	2 (3.4)
7	8 (13.6)	–
8	6 (10.2)	–
30	1 (1.7)	–
$3/4$	1 (1.7)	–
15	–	1 (1.7)
$3/5$	–	1 (1.7)
No answer	16 (27.1)	9 (15.3)

denominator, both the interaction and what was made available for the pupils' to learn in the lessons were analysed.

Pupils' experience of unit and non-unit fractions demonstrated in the lessons

The lessons were organized to engage the pupils in whole-class discussions combined with group activities with different tasks, to probe pupils' reasoning in different ways. The teachers were aware of the results from the pre-test and had carefully planned the lessons to achieve the learning goal of how to operate with unit and non-unit fractions of a discrete quantity. We analysed how the pupils seemed to experience unit and non-unit fractions of a discrete quantity in the video-recorded lessons. We suggest that pupils' difficulties with unit and non-unit have to do with how they see the role and meaning of the numerator and the denominator. The findings show three different ways that the pupils seemed to have experienced fractions during the lessons.

- 1) Some pupils see the denominator as the amount within one group, in a unit fraction, and in a non-unit fraction. For example, $1/3$ (or $2/3$) of 12 is interpreted as 3 in each group (see example 1a and 1b below).
- 2) Some pupils see the numerator as the amount within one group. For example, $2/4$ of 12 are interpreted as 2 in each group (see examples 2a and 2b).
- 3) Some pupils see a non-unit fraction to be the amount within the unit fraction. For example, the answer to $4/4$ of 8 is interpreted to be the same as for one unit of fourths ($1/4$) of 12, in this case 2 (see example 3).

The first two are related to pupils experiencing the numbers in a fraction as an amount (cf., Ball 1993; Pitkethly and Hunting 1996, p. 10). The third one, we infer as being related to the making of units. This is further described and developed in the following section. Five examples are given to illustrate what pupils seem to experience and not discern in relation to fractions, from our point of view and in this study.

Example 1a. Seeing the denominator as the amount within one group

The first example shows, as we see it, how pupils demonstrate their understanding of the denominator. In the excerpt below, two pupils seem to experience the denominator, 3, in $1/3$ of 12 as representing the amount within one group, and not the number of groups the whole is equally partitioned into. When Elsa was asked to circle $1/3$ of 12 strawberries in a picture during a whole-class discussion, she circled three of them (Fig. 3).

Excerpt 1a

- 1 Teacher C: How many is one third of these strawberries? Can someone circle one third? Elsa can you circle one third? *Elsa circles three of twelve strawberries on the overhead sheet.*
- 2 Teacher C: Is that one third?
- 3 Elsa: (inaudible)

Fig. 3 Elsa circled three strawberries to show $1/3$ of 12

4 Teacher C: Can you split it so it becomes three parts. Circle the other [strawberries] too. There should be an equal amount in each. Are you unsure?

5 Elsa: Yeah

When the teacher noticed that Elsa was unsure or answered incorrectly, another pupil (Othilia) was asked to help her. She partitioned the strawberries into three groups of four, thus solving the task correctly. The teacher, who probably wanted to check if the class could follow Othilia's solution, asked:

6 Teacher C: Are they grouped into thirds now? Have we split these strawberries into thirds now?

7 Pupils: Ah [yes]

8 Teacher C: Mm, how many are one third [of the 12 strawberries] then?

The teacher was interrupted by one pupil (Vincent), who questioned this, he said:

9 Vincent: But, there are not three in each!

Just like Elsa, Vincent wanted to have three in each group to be able to call it thirds. The teacher noticed his incorrect answer and turned to the class:

10 Teacher C: Now Vincent says there are not three in each.

11 Pupils: No

12 Teacher C: [Turning to Vincent] No, but are there [the correct answer] three in each Vincent? What are you thinking?

13 Vincent: Was it fourths?

Our interpretation is that Elsa and Vincent focused on the denominator as indicating the amount within one group (i.e. thirds as three strawberries). So, at least two pupils demonstrated this way of experiencing the role of the denominator. Thus, they did not discern the denominator as indicating the number of groups that something could be partitioned into but as showing the amount within one group. Vincent's last comment

“Was it fourths?” (line 13) supports this interpretation. In his understanding, since there are four strawberries circled in one group, it must be fourths, not thirds.

Example 1b. Seeing the denominator as the amount within one group (non-unit fraction)

The following excerpt shows another example of how pupils seem to experience the denominator as the amount within one group. However, this example is about non-unit fractions, and shows how one pupil (Sara) who was able to give a correct answer to $1/3$ of 12 but did not answer correctly to $2/3$ of 12. In the following excerpt, the teacher started with a task involving a unit fraction: $1/3$ of 12.

Excerpt 1b

- 1 Teacher A: If I say ... one third of twelve. *Twelve strawberries are posted on the whiteboard. The teacher writes $1/3$ of 12 = on the whiteboard.* What is one third of twelve? Sara? [“Is” is a literal translation of the Swedish word “är”. Note! The teacher does not use the words “how many”].
- 2 Sara: Four

Here we can see that Sara answered correctly in the task ($1/3$ of 12 is 4). However, we can see later in the interaction that Sara does not answer correctly when asked about a non-unit fraction. The teacher continued:

- 3 Teacher: ... what is two thirds of twelve? *The teacher writes $2/3$ of 12=.* If one, okay guys look at the front now, if one third of twelve is four what is two thirds of twelve. *The teacher draws two lines to separate the twelve strawberries into three groups with four in each* (Fig. 4).

It seems as if the teacher wanted the pupils to draw on their knowledge that one third of 12 is four to find two thirds of 12. However, Sara and the two other pupils demonstrated that they could not see the connection. They answered “three” and “six” although the teacher had pointed out the connection again.

- 4 Thomas: Six [in a low voice]
- 5 Teacher A: One third of twelve is four, but if one takes two thirds of twelve, what is that then? ... What were you thinking about before Mona?
- 6 Mona: Three
- 7 Teacher A: No. Sara?
- 8 Sara: Six

Fig. 4 The teacher draws two lines to separate the twelve strawberries into three groups of four

How should the answers “three” and “six” be interpreted? It is hard to know since the teacher chose not to follow up these answers in more detail. From the answer she gives, Sara does not experience two thirds as one-third taken two times. A possible explanation, in line with example 1a, is that the denominator (3) is seen as the amount in each group, and therefore six is two groups of three. Thirds could be seen as meaning three objects. In the excerpt the teacher systematically varies the numerator, $1/3$ followed by $2/3$ to make the pupils aware of the meaning of the numerator. However, $2/3$ of 12 turned out to be difficult for these pupils. To be able to operate with a non-unit fraction such as $2/3$ of 12, the role of the numerator needs to be taken into account since one needs to make a new composite unit of *two* sets of the unit fraction $1/3$. During this episode the pupils were not given an opportunity to experience that $2/3$ consists of a unit of two one-thirds.

Example 2a. Seeing the numerator as the amount within one group

The following excerpts show two more examples of how pupils experience the numbers in a fraction as representing an amount. However, in this case it is the numerator that is focused on. Thus, in the task $2/4$ of 12, the numerator is seen as indicating the amount within one group. Eva's group partitioned 12 into 6 groups with 2 in each group to show $2/4$ of 12. We make this inference from the following excerpt when one pupil (Eva) justified her solution to the operation during an oral account of a group activity in a whole-class discussion. She said:

Excerpt 2a

- 1 Eva: We split everything into two equal, into two parts. Well six, six piles with two blocks [see Fig. 5].
- 2 Teacher B: You have split it into six piles with two blocks [in each]. Let the blocks lie [still on the table] six piles. Where does it say that you should do that?
- 3 Eva: Mmm ... two, two, there, the number two with the line under [$\frac{2}{4}$]. *She points to the numerator in the fraction $2/4$ on the whiteboard.*

First Eva explained that they had split the 12 objects into “two parts”, but then corrected herself and said “six piles with two blocks”. In our interpretation, her answer “We split everything into two equal, into two parts”, (line 1) could be the

Fig. 5 Eva's group partitioned 12 into 6 groups of 2 to show $2/4$ of 12. Eva is pointing to the number 2 in $2/4$ (of 12) on the whiteboard

same as “six piles with two blocks”. The words used “two equal” or “two parts” could have been understood as ‘parts with two in each’. Hence, she had partitioned the whole, 12, into groups of 2 in each. She explained that the reason for making groups of two was “the two with the line under $\frac{2}{4}$ ” (line 3). So, in contrast to the previous examples, she seems to see the numerator as an indicator of the amount in each group and she does not to discern it as the number of sets of the unit fraction $\frac{1}{4}$ of 12.

Example 2b. Seeing the denominator or the numerator as the amount within one group

The following excerpt shows another example of how a pupil experiences the numerator as the amount within one group. However, in this case he first sees the denominator as indicating the amount within one group and later on the numerator. When Oscar was asked to circle $\frac{3}{4}$ of eight strawberries he first partitioned the whole into two parts with four in each (Fig. 6).

Excerpt 2b.

- 1 Teacher D: Have you partitioned into three fourths now?
- 2 Oscar: Aaa [unsure]..., no. I have circled two [groups].
- 3 Teacher D: You have circled two something, what have you
- 4 Oscar: [inaudible]
- 5 Teacher D: How many parts have you partitioned it into?
- 6 Oscar: Two
- 7 Teacher D: What is it called when one partition into two parts?
- 8 Oscar: Fourths
- 9 Teacher D: Nooo, every part, if you think, if you have an apple... that you have split into two parts. What is every part called? *The teacher shows an apple that they have previously split into different parts.*
- 10 Oscar: A half
- 11 Teacher D: A half. You have split it into two halves.

To the teacher’s question “What is it called when one partition into two parts?” he replied “fourths”. He seems to experience the number of objects in each part as indicating the meaning of the denominator, ‘four objects in each part’ as fourths (line 8). The teacher points out the difference between fourths and halves by contrasting

Fig. 6 Oscar partitioned 8 strawberries into 2 groups to show $\frac{3}{4}$ of 8

this task with a previous one with an apple that they had split into two parts (line 9). After this Oscar correctly partition the strawberries into four parts (line 12). He also succeeds in naming each part “one fourth” (line 13).

- 12 Teacher D: Think about if you can partition it into fourths. *The teacher wipes the board. Oscar partitions the eight strawberries into four parts.* Very good. What is every part called now then Oscar?
- 13 Oscar: One fourth.
- 14 Teacher D: Then I want to know. You can take a red pen. You are going to circle three fourths [$3/4$ of 8].

When Oscar is once again is asked to circle $3/4$ of 8 (line 14), he circles three strawberries in each part (Fig. 7). Oscar is consistent in his line of reasoning, that the numerator or the denominator indicates the amount of objects in each part.

Even if he partitioned the whole into fourths correctly, after some help from the teacher, he did not make a unit of three one-fourths ($3 \times 1/4$) to show $3/4$ of 8.

The excerpt shows that Oscar has no difficulties with partitioning into equal parts. Hence, he circles the same amount of objects in each part (see Figs. 6 and 7). This aspect of fractions he has discerned. However, he has not yet discerned the proper meaning of the numerator and denominator. Even if the teacher makes a contrast with another fraction, by comparing fourths ($1/4$) with halves ($1/2$), this appeared not to be sufficient for understanding the role of the numerator.

Example 3. Seeing a non-unit fraction as the amount within the unit fraction

This example shows, in our view, how one pupil experiences a non-unit fraction, $4/4$, as the amount within one of the four equally partitioned groups. In the following excerpt, the teacher drew eight strawberries on the board and started with a unit fraction task, $1/4$ of 8.

Excerpt 3

- 1 Teacher A: If one fourth of eight is two. What is then two fourths of eight? *The teacher writes $2/4$ of 8=.* One fourth of 8 is two, what is then two fourths of eight? Joanne?

Fig. 7 Oscar partitioned 8 strawberries into 4 groups to show fourths. When asked again about $3/4$ of 8 he circled three in each group

- 2 Joanne: What is two four [stops repeating two fourths] .. it is four.
The teacher writes 4.
- 3 Teacher A: What is three fourths of eight? ... Frida? *The teacher writes $3/4$ of $8=$*
- 4 Frida: [inaudible]
- 5 Teacher A: Here is one fourth *The teacher points to the first column of two strawberries from the right hand side.*
here is two fourths *The teacher points to the second column of strawberries from the right hand side.*
and there three fourths *The teacher points to the second and third column from the right hand side.*
- What is three fourths of eight Felicia ... what did you say I just didn't hear what you said.
- 6 Frida: Six

Here we can see that two pupils, Joanne and Frida (lines 2 and 6), managed to answer correctly in the tasks with non-unit fractions, although this came about with some support from the teacher pointing to sets of fourths (line 5). In the excerpt below, we can see that another pupil (John), experiences $4/4$ of 8 in another way, despite the examples given previously. To the task $4/4$ of 8 he answered “two” (line 8).

- 7 Teacher A: Three fourths of eight is six and four fourths of eight is ... John?
- 8 John: Two [a low voice]

The answer “two” indicates that John does not experience that $4/4$ includes four sets of $1/4$, despite the fact that the teacher has shown that when the sets are added the correct answer ($1/4 + 1/4 + 1/4 + 1/4 = 4/4$) is generated. John's answer “two” could be interpreted to mean that he focuses on the amount within one set of $1/4$ of 8, the fourth.

John does not appear to see $4/4$ of 8 as 4 sets of $1/4$ of 8 but as one set of $1/4$ instead. Why? One explanation could be the use of words. It is possible that John saw $4/4$ as ‘the fourth part’ of fourths [in Swedish ‘fjärde delen’] and not as four ‘fourths’ [in Swedish ‘fjärdedelar’] (see Fig. 8). In Swedish one fourth (singular) is called “fjärdedelen” and ‘fourths’ (plural) is called “fjärdedelar”, and the difference between

Fig. 8 A possible explanation of John's answer “two”, as the ‘fourth part’ of strawberries starting from the right hand side

one fourth and ‘the fourth’ part [fjärde delen] in Swedish is only heard in the intonation of the words, they are pronounced in a very similar way. His answer, 2, fits with this logic, since there are 2 strawberries in the fourth group from the right hand side. This interpretation is partially supported by the teaching in this episode, when the teacher pointed to the different sets of $1/4$, for instance when she said “here are two fourths” and pointed only to the second column of strawberries from the right hand side (line 5). We make this inference with support from the following excerpt. When John did not answer the question “*What is $4/4$ of 8?*” correctly, the teacher continued discussing with him. She focused on the unit fraction $1/4$ of 8 (line 9) and pointed (partly) to two sets of $1/4$ to show $2/4$ of 8. However, now John’s answer to “*What is $4/4$ of 8?*” is “four” (lines 10 and 12).

- 9 Teacher A: John, one fourth is two. *Points to the first set of $1/4$ from the right hand side. Then she points to $2/4$ of 8 written on the board. Two fourths are those two. The teacher points to the top two rows of strawberries (not the two whole sets).*
- 10 John: Four.
- 11 Teacher A: Three fourths ... it is those two, *The teacher points to the second and the third set from the right hand side, and four fourths John, how many is that? The teacher sweeps her hand over the four strawberries on the top of the sets (not the whole four sets).*
- 12 John: Four

In our interpretation, the teacher was not explicit about the fact that, for instance, $2/4$ of 8 includes 2 sets of $1/4$, or that $3/4$ includes all 3 sets of $1/4$. The excerpt shows that John does not discern $4/4$ as a composite unit of 4 sets of $1/4$. This could explain why John answered “two” and “four” to the question “What is $4/4$ of 8?”.

Discussion and conclusion

In this article we have explored pupils’ learning from the point of view of aspects of fractions not discerned by pupils in the lessons. Variation theory has been used as an analytical framework for analysing the data. This approach not only revealed pupils’ learning difficulties but also identified what needed to be learned – the critical aspects – to be able to understand unit and non-unit fractions of discrete quantities. The present study confirms previous research that shows that pupils seem to find operating with non-unit fractions of a discrete quantity more difficult than operating with unit fractions (cf., Kieren 1993). Although some progress was made between the pre- and post-tests, less than half of the pupils were able to solve tasks with non-unit fractions after two lessons, compared to about 85 % who successfully operated with unit fractions. The test results show that some incorrect answers were more common than others, and these answers indicate specific ways of understanding. One incorrect way of answering increased after teaching, the answer two (to $3/5$ of 10 and $2/3$ of 6). This answer coincides with the amount in one set, i.e. $1/5$ of 10 or $1/3$ of 6. Why did this answer become more frequent after the lessons? A conjecture is that it could be related to the teaching. For instance, if an example where the number of the numerator (e.g. $2/3$ of 6) coincides with the amount in each set (of $1/3$, which is also 2), is used, pupils could interpret this to mean that the

numerator indicates the amount in each set (Petit et al. 2010). This shows the importance of carefully considering the particular examples when teaching to enable the variation to be clearly discerned by the pupils and to avoid the distraction created when the same numbers occur in an example with different meanings. Another example of how the teaching could have affected the way the pupils understood fractions is given in excerpt 3. Here the teacher pointed only to “the second” set of $1/4$ when talking about $2/4$ of 8. This could explain why John, who answered 2 when asked about what $4/4$ of 8 was, interpreted $4/4$ as not including all four sets of $1/4$ (see excerpt 3, line 5). In this case, it is possible that linguistic usage could have contributed as well. This is discussed further below.

Earlier studies suggest that pupils’ difficulties with non-unit fractions of discrete quantities could be related to their experience of the continuous model, for instance $3/5$ is seen as 3 pieces of a 5-piece pizza (Pitkethly and Hunting 1996). From this follows that, when operating with fractions of a discrete quantity, some pupils interpret a fraction as two independent numbers, “ a/b ” as meaning “ a things out of b things” (Thompson and Saldanha 2003). However, it does not explain, on a fine-grained level, what particular pupils notice and discern, or are unaware of, during classroom teaching when encountering non-unit fractions. According to the theoretical framework adopted, learning is viewed as the discernment of critical aspects of what is learned (Marton and Booth 1997). Marton and Pang (2006) argues that, to be able to discern a phenomenon in a specific way, the learner must differentiate, make finer and finer distinctions and become attuned to distinguishing aspects, or critical differences (Marton and Pang 2006, p. 28). Our findings lead us to suggest the learning difficulties the pupils in this study had with unit and non-unit fractions of discrete quantities could be explained by dimensions of fractions not discerned or differentiated between. Our analysis of lessons (excerpts 1a, 1b, 2a, 2b) show that pupils seem to experience the figures of the numerator or denominator as indicating the amount within each group of the partitioned whole. For instance, some pupils experienced $2/3$ of 12 as ‘groups with 3 in each’ (focus on the denominator) or ‘groups with 2 in each’ (focus on the numerator), and not as two sets of $1/3$ of 12 (with 4 in each) (Ball 1993; Empson and Levi 2011). Hence, it is likely that these pupils had not differentiated between the number of groups the whole should be partitioned into and the amount within each group. We also found that some pupils could experience a non-unit fraction, e.g. in $4/4$ of 12, as the amount within one set of $1/4$. Hence, it is possible that they may not have discerned that $4/4$ of 12 include all four sets of $1/4$ and does not differentiate between, in this case, ‘the fourth’, and ‘fourths’. If pupils are to learn to operate with fractions, it is of fundamental importance that they can differentiate between the roles of the numerator and denominator. However, in order to give a correct answer to a question about a unit fraction, for instance, $1/4$ of 12, it is sufficient to operate with the denominator only. Ball (1993) argues that to understand what $1/4$ means, one need only divide the whole into four parts. But, non-unit fractions require complex compositional thinking, that is $3/4$ entails both dividing into four parts and multiplying the result by 3. Hence, it is not necessary to discern the numerator in $1/4$ of 12 to get the correct answer, whereas with $2/4$ of 12 both the numerator and the denominator must be taken into account and they must be taken into account at the same time. So, when operating with a non-unit fraction, it is necessary to focus on the numerator and the denominator simultaneously and to discern their different meanings. In the case of unit fractions, this differentiation is not significant in the same way. We

would suggest that one critical aspect of learning to operate with non-unit fractions of discrete quantities is to be aware of and learn this difference. That this critical aspect was not discerned is illustrated, we think, by Sara (excerpt 1b). She was able to solve $1/3$ of 12, but not $2/3$ of 12. That the pupils might not have discerned this critical aspect could explain some of the differences between the results of the pre- and post-tests. Consequently, a pupil was able to solve an operation with a unit fraction as a division without taking the numerator into account and without needing to make a new composite unit, in contrast to the case with $2/3$ of 12. From this follows that when pupils have worked only with unit fractions, they may not have been confronted with the essentially compositional nature of fractions, which must be understood when working with non-unit fractions (Ball 1993; Empson and Levi 2011).

In conclusion, the teachers at each of the schools in this study collaborated when planning the lessons and enacted the planned lessons in their classes. They had used information about what had previously been found to be critical for pupils' learning in a learning study. Hence the teachers were aware of the importance of understanding the roles of numerator and denominator and planned the lessons accordingly. The teachers used ideas from variation theory and systematically varied one aspect of fractions at a time to make the aspect noticeable for the pupils. Even so, many pupils did not learn what was intended. We advocate that some of these pupils did not have the opportunity to experience critical aspects of fractions. This shows once more that the teaching and learning of fractions are complex enterprises, and that some pupils do not learn even when their teacher has deliberately planned to overcome difficulties. *This study adds to previous research by identifying and viewing pupils' difficulties from the point of view of what dimensions of fractions the learners have or have not discerned or differentiated between* (cf., Olive and Vomvoridi 2006). Within variation theory learners need to discern critical aspects in order to learn something specific, in this case operating with non-unit fractions of a discrete quantity. For a pupil to discern the critical aspects these need to be addressed during lessons and highlighted by the teacher or other pupils (Kullberg et al. 2009; Mason 2010).

As is reported in the literature equi-partitioning is seen as a key for learning about fractions (Empson and Levi 2011). We found it interesting that the pupils in this study did not appear to have difficulties with partitioning into equal parts (see excerpt 2a, 2b). For example, when Oscar was asked to circle $3/4$ of 8 he first partitioned the objects into two groups with four objects in each. After some help from the teacher he correctly partitioned the whole into four groups with two objects in each. Oscar had already mastered the rule that there should be an equal number of objects in each part. When asked to circle $3/4$ again, he circled three objects in each group (excerpt 2b). Judging from the data in this study the difficulty seemed instead to be *how many parts to partition into, how many objects there are in each part* (excerpt 1a, 1b, 2a, 2b) and *the making of new units* (excerpt 3) simultaneously.

To conclude, in this study we have not just shown that the pupils had difficulties with numerator and denominator and how they understand these, but that these difficulties come from not having discerned aspects which probably were necessary to discern, and that the pupils were not provided the opportunity to discern them either. In the documentation of the learning study in Hong Kong the teachers had described some critical aspects. What they had found pupils needed to discern, this was described as "being aware of the role of the numerator and denominator". However, from our analysis we

find this description was too general to provide a clear idea of what the pupils must learn in order to understand the role of the numerator and denominator and how that could be taught. We would suggest that it was critical for pupils' learning to:

- differentiate between the number of groups the whole should be partitioned into (the denominator) and the amount within each group
- differentiate between the amount of units (the numerator) and the amount of objects in each unit
- differentiate between 'the fourth' and (four)'fourths'

This result, we think, could give teachers a guide how to teach the meaning of the numerator and denominator. Or put differently, the key is to make it possible for the pupils to make distinctions between certain aspects of the numerator and denominator.

Finally, we would suggest that variation theory helped us to view the learners' perspective in lessons by pointing out what they needed to experience and differentiate between in order to learn and overcome their difficulties.

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