GÖTEBORGS UNIVERSITET
INST FÖR PEDAGOGIK, KOMMUNIKATION OCH LÄRANDE

Mathematics textbooks for teaching

An analysis of content knowledge and pedagogical content knowledge concerning algebra in mathematics textbooks in Swedish upper secondary education

Licentiatuppsats i ämnesdidaktik, inom ramen för forskarskolan CUL

Wang Wei Sönnerhed

2011-06-15

To *Stefan, Alexander, Olivia*

Abstract

In school algebra, using different methods including factorization to solve quadratic equations is one common teaching and learning topic at upper secondary school level. This study is about analyzing the algebra content related to solving quadratic equations and the method of factorization as presented in Swedish mathematics textbooks with subject matter content knowledge (CK) and pedagogical content knowledge (PCK) as analytical tools. Mathematics textbooks as educational resources and artefacts are widely used in classroom teaching and learning. What is presented in a textbook is often taught by teachers in the classroom. Similarly, what is missing from the textbook may not be presented by the teacher. The study is based on an assumption that pedagogical content knowledge is embedded in the subject content presented in textbooks. Textbooks contain both subject content knowledge and pedagogical content knowledge.

The primary aim of the study is to explore what pedagogical content knowledge regarding solving quadratic equations that is embedded in mathematics textbooks. The secondary aim is to analyze the algebra content related to solving quadratic equations from the perspective of mathematics as a discipline in relation to algebra history. It is about what one can find in the textbook rather than how the textbook is used in the classroom. The study concerns a teaching perspective and is intended to contribute to the understanding of the conditions of teaching solving quadratic equations.

The theoretical framework is based on Shulman's concept pedagogical content knowledge and Mishra and Koehler's concept content knowledge. The general theoretical perspective is based on Wartofsky's artifact theory. The empirical material used in this study includes twelve mathematics textbooks in the mathematics B course at Swedish upper secondary schools. The study contains four rounds of analyses. The results of the first three rounds have set up a basis for a deep analysis of one selected textbook.

The results show that the analyzed Swedish mathematics textbooks reflect the Swedish mathematics syllabus of algebra. It is found that the algebra content related to solving quadratic equations is similar in every investigated textbook. There is an accumulative relationship among all the algebra content with a final goal of presenting how to solve quadratic equations by quadratic formula, which implies that classroom teaching may focus on quadratic formula. Factorization method is presented for solving simple quadratic equations but not the general-formed quadratic equations. The study finds that the presentation of the algebra content related to quadratic equations in the selected textbook is organized by four geometrical models that can be traced back to the history of algebra. These four geometrical models are applied for illustrating algebra rules and construct an overall embedded teaching trajectory with five sub-trajectories. The historically related pedagogy and application of mathematics in both real world and pure mathematics contexts are the pedagogical content knowledge related to quadratic equations.

Keywords: mathematics textbooks, school algebra, solving methods, factorization, solving quadratic equations, mathematics teaching, content knowledge, pedagogical content knowledge, geometrical models, algebra history, embedded teaching trajectories

Contents

Appendixes

1. The twelve mathematics textbooks used for the third round of analyses
2. An example of one of the twelve tables containing coded content elements
3. The introduction activity on page 7 in the textbook of *Matematik 4000* B

Acknowledgements

In my work on this thesis, I have received great help from many people. From the bottom of my heart, I would like to thank those who have helped and encouraged me in the process of my thesis writing. I am especially grateful to my supervisors Professor Berner Lindström and Fil. Dr. Jonas Emanuelsson for all theoretical teaching and guidance when I entered the research community from a laymanship. I thank you for being patient and supporting me with our common project. Without your professional support and the project, I would not have been able to complete this thesis.

This thesis has been seen help in many ways. My friend and colleague Cecilia Kilhamn has engaged in my thesis with great enthusiasm. You have helped me to make a clear construction of the thesis and understand many theories; in particular, you encouraged me all the time when I had difficulties. Besides, you have helped me with the proofreading of my language through the whole thesis! I here also thank Angelica Kullberg who shared many theoretical discussions with me and encouraged me to carry on my study. Ingemar Holgersson has given me valuable opinions on the part of mathematics theory. Laura Fainsilber has read and commented the theoretical part of algebra. Thomas Lingefjärd has shown great interest and support to my thesis. Johan Häggström gave me a good introduction to research of algebra teaching and learning. Marianne Dalemar provided great help when I needed research articles at NCM library. I would also like to give many thanks to Åse Hansson and Maria Reis from FLUM group and Professor Ulla Runesson who encouraged me and gave me their opinions in my work.

Furthermore, I would like to express my thanks to the graduate school CUL (Center for Educational Science and Teacher Research) and the Faculty of Education at the University of Gothenburg who gave me financial support. Many thanks to Klas Ericson and Marianne Andersson who helped me with practical things during my doctoral study. I would also like to thank Swedish publisher Natur & Kultur who provided information on the investigated mathematics textbooks.

At last, I thank my beloved family; Stefan, Alexander and Olivia for always being there for me. Thank you Stefan for your love and lots of encouragement as well as your language support. Without you and our children, I would not have been able to finish this thesis!

Gothenburg, April 28, 2011

Wang Wei Sönnerhed

1. Introduction

1.1 Personal interests

Being an immigrant researcher, I received my upper secondary mathematics education in the People's Republic of China. During the 1980s, China was not open enough to be able to communicate with the outside of world within the field of mathematics education. Mathematics teaching at that time was characterized by manipulating algorithm steps without the use of calculators or computer programs. Reasoning about the solution of a problem was a common teaching approach in mathematics teaching in my school. When I moved to Sweden, I repeated my mathematics education at Swedish upper secondary school and later started my teacher training within the field of mathematics education at Jönköping University. During that time, I was offered opportunities to do my teaching practice in an upper secondary school where I found that teaching to solve quadratic equations by using the quadratic formula was the essential method. Another method, like factorization, was not in focus. I wondered why the factorization method was not emphasized in teaching solving quadratic equations. This finding was not only generated from the teaching practice but also from my own experiences of studying mathematics at a Swedish upper secondary school and at university. Algebra teaching concerning how to solve quadratic equations was different from my Chinese educational background, in which I was taught to use factorization as an essential method to solve a quadratic equation. The quadratic formula was regarded as a secondary tool in order to deal with the quadratic equations that were difficult to be solved by factorization.

The different teaching focuses regarding the topic of solving quadratic equations made me curious about what mathematics is taught and why it is taught differently. With these questions in my mind, I sought answers. In 2006, I was accepted as a doctoral student by a graduate school (the Center for Educational Sciences and Teacher Research) at Gothenburg University. I am very grateful to the graduate school for providing me with such an opportunity to do research within the field of teaching and learning school algebra. Without hesitation, I started by searching for previous research related to teaching quadratic equations and factorizations. To my surprise, I have found that I am not alone being interested in these topics. Mathematics educators from other countries like Singapore, Thailand, Canada, and the USA are interested the same topics. As a result of my research review, I have realized that different teaching focuses concerning solving quadratic equations depend on what mathematics teaching culture these come from. When related to quadratic expressions, the factorization method is emphasized in algebra teaching as a common topic in other countries as mentioned above (Bossé & Nandakumar, 2005; Nataraj & Thomas, 2006; Vaiyavutjamai & Clements, 2006; Zhu & Simon, 1987). Based on this background, I wondered how Swedish mathematics education handled the mathematical topics like quadratic equations and factorization at upper secondary school level, and what the teaching of such topics is like. Guided by my interests in algebra knowledge and teaching, I needed to analyze empirical material that could cover both fields and at the same time represent Swedish mathematics culture. Therefore, I decided to investigate Swedish upper secondary mathematics textbooks since textbooks themselves are used as important teaching resources at school and contain subject knowledge and pedagogical functions. My study focuses on analyzing algebraic content as it is presented in the mathematics textbooks.

1.2 Subject content and teaching the subject

Research on teaching has in the past been focused on teaching effects related to student achievement (Floden, 2001). Most of the research up until the mid-1970s was based on looking for associations between measuring students' learning achievements and variables from classrooms such as teachers' actions in process-product research. Up until 2001, process-product research was still a major stream of work, but it declined in the late 1970s. However, there are different answers for the question on the effects of teaching associated with students' achievements. Lee Shulman (Floden, 2001) suggests a shift to teacher's subject matter knowledge but encourages all different research approaches that could be used in order to improve student performance and learning. Among these, research on examining whether the changes of teaching materials can lead to improvement in students' learning has being expanded to complete the effects of a teaching paradigm. Policy studies use methods like examining teaching and learning content as well as how much time students spend on learning a particular content. Researchers may analyze the content of textbooks, tests or other material and compare it with teachers' instructions recorded by observations, interviews and teacher logs. The shift from teachers' qualification and education to teachers' subject matter knowledge since Shulman's article in 1986, has made research focus on teaching content, instruction and curriculum materials (Floden, 2001).

The common object in research about teaching and learning mathematics is the mathematical content. Teaching specific mathematical content requires teachers' *pedagogical content knowledge – PCK* in short (Shulman, 1986b) – which links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, teachers need to know things like what topics students find interesting or difficult or the representations most useful for teaching a specific content idea. Such knowledge intertwines aspects of teaching and learning with contents. It is built up by teachers over time as they teach special topics to students or by researchers as they investigate the teaching and learning of specific mathematical ideas (Ball & Bass, 2000). In this case, content knowledge goes beyond mathematical content and it involves pedagogical content knowledge and curricular knowledge (Shulman, 1986b). It is common that PCK is studied within the fields of teachers' knowledge and teaching. Following the Shulman's argument, it should be important to investigate how PCK is built in teaching and learning materials.

With its departure from Shulman's original concept, this study is about investigating possible ways of teaching school algebra by analyzing algebra content in mathematics textbooks. In the research of school algebra, Kieran (2007) points out that algebra learning has been studied more than algebra teaching. Algebra content knowledge has been widely studied and covered many areas such as equations, algebraic expressions, algebraic operational rules, simplifying algebraic expressions, problem solving, modeling and so on. Among these areas, quadratic equations are less studied at upper secondary school level. Algebra is often regarded as a difficult area for Swedish students in mathematics studies according to the international test results of TIMSS and PISA (Häggström, 2008). Some of the latest studies related to classroom discourse have been carried out in this field, from lower to upper secondary schools in a Swedish context (Kilhamn, 2011; Olteanu, 2007; Persson, 2010). They have enriched research on algebra learning within the areas of negative numbers and conceptual understanding of algebra symbols. Based on this background, this study aims at contributing to the research of school algebra through studying mathematics textbooks – one of the influential factors related to algebra teaching.

More precisely, the objective of this study is to study the subject of algebra related to factorization and solving quadratic equations in order to uncover the embedded PCK built into mathematics textbooks concerning these algebraic content. In Swedish classrooms, students spend a substantial part of lesson time on using mathematics textbooks (Johansson, 2006). As artifacts (Wartofsky, 1979) and as a major resources for teaching and learning, mathematics textbooks often cover topics presented by teachers in classrooms. However, topics which do not appear in textbooks are not likely to be presented by teachers (Johansson, 2006). "Teaching of the text has always been the teacher's primary function, with the teacher as mediator" (Pepin, Haggarty, & Keynes, 2001, p. 7). The textbook can assist inexperienced teachers in deciding what to teach and also in keeping students work at the same pace (Selander, 2003). The pedagogical function makes textbooks teaching aids. The essential role of mathematics textbooks in Swedish mathematics classrooms is an important background to this study on analyzing mathematics textbooks from a teaching perspective.

This study intends to discover pedagogical content knowledge by means of looking for embedded teaching trajectories related to algebra content concerning quadratic equations. This is based on an assumption that textbooks have embedded teaching trajectories to present subject content according to certain orders. The term of *embedded teaching trajectory* in this study derives from the expression of *a hypothetical learning trajectory* used by Paul Cobb (2001). A hypothetical learning trajectory includes both a possible learning route or trajectory with important mathematical ideas and the specific actions that might be used to support and organize learning along the envisioned trajectory according to Cobb (2001). The envisioned trajectory is hypothetical in the sense that it embodies hypotheses about what might be possible for students' mathematical learning in a particular domain (Cobb, 2001). The point here is that the hypothetical learning trajectory is imagined rather than how it is manifested. A *teaching trajectory* concerning a special subject involves a mathematics goal and a teaching path or developmental progression along which students are expected to learn the subject and develop their mathematical competences (Clements & Sarama, 2009). Combining the meaning of hypothetical trajectory and teaching trajectory, I use the term of *embedded teaching trajectory* to refer the possible teaching paths with specific mathematical goals built into a mathematics textbook. The specific mathematical goal analyzed in this study is to teach how to solve quadratic equations. Therefore the content analysis of this study is to examine algebra content related to quadratic equations in the textbook by using the concepts *content knowledge* CK (Mirshr & Koehler, 2006) and *pedagogical content knowledge* PCK (Shulman, 1986b) as a theoretical framework in order to find the embedded teaching trajectory.

1.2 The aim of the study

The aim of this study is primarily to explore what pedagogical content knowledge regarding algebra, in particular quadratic equations, is embedded in the mathematics textbooks used for Swedish upper secondary schools. The study relates to both algebra content and pedagogical content knowledge in the textbooks. An important step on the way – and a secondary aim of my study − is to analyze the algebra content presented in the textbooks. This will be done from the perspective of mathematics as a discipline and especially in relation to the historical development of algebra as a field of knowledge. It is about what one can find in the textbook rather than how the textbook is used in the classroom. This study reflects an analytic interest of algebra content knowledge as subject matter content knowledge. In order to combine these two aims, I use the CK-PCK framework to analyze the algebra content in the textbook. The study is intended to contribute to the understanding of the conditions of teaching solving quadratic equations.

Research questions are formulated below:

1. What mathematics do Swedish upper secondary mathematics textbooks reflect in their presentations of quadratic equations?

To answer the first question the following detailed questions were posed:

a) What algebra content related to quadratic equations is presented in the textbooks?
b) In which order is quadratic equations and functions presented and do they have connections to each other?
c) What is the most emphasized method for solving quadratic equations presented in the textbooks?
d) How is factorization presented in the textbooks?

The results of the first research question set up a basis for the main in-depth analysis of one textbook, in order to answer research questions two.

2. What aspects of pedagogical content knowledge can be traced in the way a Swedish upper secondary school textbook presents the algebra content related to quadratic equations?

To answer the second question, I analyzed the mathematical texts, examples, activities and exercises of the textbook in detail. Then the following questions were posed:

a) How is mathematical content presented or explained?
b) What is the character and function of the presented examples and exercises?
c) What embedded teaching trajectories are built into the presentations of quadratic equations in the textbook? How are those trajectories constructed?

1.3 Structure of the thesis

In this part, I will present the organization of the whole thesis. Since it is based on a combined CK-PCK (Mishra & Koehler, 2006; Shulman, 1986b) theoretical framework, the thesis will be organized according to the two aspects of the framework.

The second chapter begins with a theory of artifact (Wartofsky, 1979) as a general perspective of this study and then presents the overall theoretical framework for the study: content knowledge and pedagogical content knowledge (Mishra & Koehler, 2006; Shulman, 1986b), and mathematical representations (Goldin, 2008; Vergnaud, 1987). A short review on mathematics application in mathematics education (De Lange, 1996) is also carried out in this chapter. The aim is to show the connections between these theoretical aspects and the study.

The third chapter presents mathematical content relevant for this study. The chapter contains three parts: a review of algebra history related to elementary algebra, three approaches to solving quadratic equations, and factorization related to abstract algebra. The algebra content in this chapter is the core content in the whole study as the subject matter content knowledge

(CK) presented in the textbooks. The aim is to seek the historical link and relation to mathematics discipline in regard to the algebra content presented in the textbooks.

The fourth chapter consists of two reviews of previous studies in the fields of textbook research and school algebra research. Within the area of textbooks, the question why this study relates to teaching is answered; two surveys of textbook research are summarized, and a review of previous studies on mathematics textbook research is presented. A conclusion of the textbook research field will be drawn after the first review. Within the area of algebra teaching and learning, previous research on algebra teaching and learning in general will be reviewed. Previous studies on teaching and learning factorization and solving quadratic equations related to this study will be presented. A conclusion of teaching and learning algebra will be drawn after the second review. The aim is to get an insight into the research in the two fields: textbook research and school algebra research in order to position my study in these two fields.

In the fifth chapter, the research method and process will be presented. I will first introduce content analysis as the research method to this study, and then focus on presenting the analyzing process containing four rounds of analyses of the investigated mathematics textbooks. Afterwards, the analytical tools used for analyzing mathematical texts and exercises in the textbook will be presented. Finally, I will reason about the reliability of this content analysis.

The sixth chapter presents the results derived from the analyses of the investigated mathematics textbooks by relating to the research questions.

The seventh chapter concludes the thesis by discussing the findings and possible implications of the findings for teaching algebra related to solving quadratic equations, as well as points of interest for the future research. Discussion will be carried out in relation to early research within the area of algebra content.

2. Theoretical Perspective

This chapter elaborates on the theoretical concepts used in this thesis and they include: artifact; pedagogical content knowledge; content knowledge; mathematical representations and applications.

2.1 Basic understanding of textbooks

This study makes use of Wartofsky's idea of *artifact* (1979) as a general perspective. As an education tool, a textbook can be regarded as an artifact (Johansson, 2006). Artifacts are tools made by human beings in actions and are applied by (or function as) humans according to different needs. Artifacts are the production or reproduction of human beings' social activities (Wartofsky, 1979). The essential character of the artifact is that "its production, its use, and the attainment of skill in these, can be transmitted, and thus preserved in a social group, and through time, from one generation to the next" (Wartofsky, 1979, p. 201).

Artifacts according to Wartofsky (1979), have characteristics of representations and reflexive sense in human perception. Human actions for survival and development in social activities throughout history were transmitted and preserved in the forms of symbols and images reflecting these actions. Wartofsky categorized artifacts into three kinds. The first of these three kinds are primary artifacts, which are directly used in the production of human activities (e.g. tools like axes, needles, bowls; modes of social organization; bodily skills and technical skills in the use of tools). Secondary artifacts are representations of the modes and skills that human beings have used in the production. Tertiary artifacts are "abstracted from their direct representational function" (Wartofsky, 1979, p. 209). They constitute free constructions in the forms of rules and operations from the actual physical world. They are derived from human perceptions of the historical actions but no longer bound to them, but at the same time they embody the objectification of human knowledge and intention, according to Wartofsky (1979).

Science theories, books, texts and so on can be regarded as tertiary artifacts since they can mediate and transform the embedded meanings and influence the world (Säljö, 2007). In my study, mathematics textbooks are analyzed as tertiary artifacts (Wartofsky, 1979) since the algebra content presented in textbooks are human products and entail mathematical knowledge perceived by human beings. The mathematical knowledge in the textbook is taught and learned in order to make learners prepared for the actual or future use of mathematical tools.

2.2 Content Knowledge and Pedagogical Content Knowledge

The analytical tools in this study are developed from theoretical concepts advocated by Shulman (1986b) and his followers Mishra and Koehler (2006). Traditionally, pedagogical knowledge and content knowledge are treated separately in teacher education (Mishra & Koehler, 2006). In the introduction of this thesis, it is mentioned that the focus of research on teaching has shifted from teacher behavior and process-product studies to teaching and teaching content since Shulman declared a new paradigm – pedagogical content knowledge (Floden, 2001).

What Shulman (1986b) emphasizes is the importance of the teaching content. Shulman argues that researchers can not ignore one central aspect of teaching in the classroom: the subject matter. This includes how the subject matter is transformed from the knowledge of a teacher into the content of instruction and how particular formulations of that content relate to what students come to know or misinterpret. This subject matter research has been absent from the studies of teaching and is called the "missing paradigm" problem, although historically teaching competence involved both knowledge of pedagogy and content. The subject matter relates to the content of the lessons taught, the questions asked, and the explanations offered according to Shulman (1986b).

Shulman (1986b) advocates that teacher's *content knowledge* involves teaching factors and the organization of them in a teacher's mind. The teaching factors are for example: the teacher's understanding of the subject; the sources of the teacher's knowledge such as textbooks, subject literature, teaching material and so on; organization of subject material; the teacher's knowledge of students and their learning; the teacher's knowledge of curricula etc. He has defined three categories of content knowledge: *subject matter content knowledge*, *pedagogical content knowledge* and *curricular knowledge*.

Subject matter content knowledge

A teacher's understanding of subject matter content knowledge requires his or her understanding of the facts and concepts of the subject content and the structures of the subject. Drawing on Schwab (1978), Shulman (1986b) indicates that the structures of a subject include both the substantive and the syntactic structures. The substantive structures refer to ways of organizing the basic concepts and principles of the discipline in order to relate them to the facts. The syntactic structures refer to a set of ways to establish truth, falseness, validity or invalidity of discipline–like rules, in other words a grammar. Shulman (1986b) argues that a teacher should be able to define and explain the theory of a discipline as well as relate the theory to teaching. The teacher should also be able to judge what is important and less important in this discipline in relation to curricula.

What Shulman emphasizes concerning *subject matter content knowledge* is the concepts, principles and rules of the discipline in a subject. There are different understandings of the terms produced from Shulman's work. Mirshra and Koehler (2006) regard subject matter content knowledge as *content knowledge* - "knowledge about the actual subject matter that is to be learned or taught" (p. 1026). Mishra and Koehler (2006) agree with Shulman, pointing out that content knowledge includes "knowledge of central facts, concepts, theories, and procedures within a given field; knowledge of explanatory frameworks that organize and connect ideas; and knowledge of the rules of evidence and proof" (p. 1026).

Pedagogical content knowledge

Pedagogical content knowledge "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986b, p. 9). What Shulman means here is not the pedagogical knowledge of teaching in general, such as classroom organization and management. PCK merges subject matter content knowledge with pedagogical knowledge according to Shulman:

> …the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of

> representing and formulating the subject that make it comprehensible to others. […] Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (Shulman, 1986b, p. 9).

Mishra and Koehler (2006) interpret this passage and point out that successful teachers would have to confront both content and pedagogy issues at the same time by adapting the aspects of content most relevant to its teachability. They emphasize that the core of PCK is that subject matter is transformed for teaching. This is done when the teacher interprets the subject matter and finds different ways to represent it and make it accessible to learners. For Mishra and Koehler, PCK represents the blending of content and pedagogy into an understanding of how particular aspects of subject matter are organized, adapted, and represented for instruction.

Similar to Shulman's PCK, Mishra and Koehler (2006) have developed their PCK which includes knowing what teaching approaches that fit the content and how elements of content can be arranged for better teaching. Their PCK "is concerned with the representation and formulation of concepts, pedagogical techniques, knowledge of what makes concepts difficult or easy to learn, knowledge of students' prior knowledge, and theories of epistemology" (p. 1027). They emphasize the knowledge of teaching strategies related to appropriate conceptual representations in order to address learner difficulties and misconceptions and fostering the learner's meaningful understanding.

Originally, Shulman's PCK included many more categories such as curriculur knowledge and educational context. Curricular knowledge according to Shulman (1986b) refers to the teacher understanding and being familiar with the curriculum and various instructional materials designed for the teaching of particular subjects and topics at a given level available in relation to curriculum programs.

Mishra and Koehler make the distinction between PCK and CK without involving curricular knowledge though their idea is consistent with Shulman's idea.

Since Shulman founded theoretical concept of PCK, thousands of articles, chapters in books and reports have studied the notion of pedagogical content knowledge in various subject areas (Ball, Thames, & Phelps, 2008). A research survey done by Ball et al. (2008) shows that about one fourth of the articles on pedagogical content knowledge are in science education with fewer in mathematics education. The field has still made little progress on developing a coherent theoretical framework for content knowledge for teaching. It lacks a clear definition of pedagogical content knowledge according to Ball et al. (2008). The qualitative study carried out by Ball et al. has its point of departure in teaching instead of teachers. They seek to develop a practice-based theory of mathematical knowledge entailed by and used in teaching. They emphasize what teachers must know in order to carry out the teaching. Their focus is both on "pure" mathematics from a disciplinary knowledge point of view and the practical world of teaching. Their empirical result suggests that content knowledge for teaching includes many aspects and that existing theoretical frameworks need refinement. They divide *subject matter knowledge* into three domains: common content knowledge (CCK), horizon content knowledge (HCK) and specialized content knowledge (SCK); and *pedagogical content knowledge* into three other domains: knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC). CCK is defined as the mathematical knowledge that is commonly known among those who know and use mathematics outside of teaching. CCK is not unique to teaching. SCK is the mathematical knowledge and skill unique to teaching. Having SCK, teachers are familiar with students' errors and have approaches to work out the problems and therefore make teaching effective (Ball et al., 2008). In the KCT domain, teachers design instructions with chosen mathematical tasks and set sequences for particular content, for example: which examples are used in the beginning of the sequence for the particular content and which ones will be used to take the students deeper into the content? Such questions require "an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al., 2008, p. 401). The refinement of PCK and CK made by Ball et al (2008) has emphasized the detail level of the components of these two categories such as CCK and KCT.

To sum up, Shulman (1986b) regards subject matter content knowledge, pedagogical content knowledge and curricular knowledge all together as teacher's content knowledge while Mirshra and Koehler (2006) make a distinction between PCK and CK without involving curriculum knowledge. Ball et al. (2008) categorize PCK and CK with the refined sub-domains. In general, PCK intertwines subject content knowledge with pedagogical knowledge. It is about knowledge for teaching subject content. PCK involves different ways representing and organizing a subject matter in order to make it accessible to learners. It concerns the discipline in a subject and pedagogy at the same time. It also involves teacher understanding of the subject matter and knowledge of students and curriculum. CK includes concepts, principles, rules, and theories of the discipline in a subject.

Well aware of researchers trying to make distinctions between Shulman's PCK categories, I combine Shulman's PCK with Mishra and Koehler's CK concept to form a theoretical framework *CK-PCK* as the analytical tools in my study. With this tool I analyze how the algebra knowledge in the textbook reflects the historical development of algebra as a discipline of mathematics. This includes algebraic concepts, theories, rules and procedures as well as mathematical proof. Analyzing how algebra knowledge is presented, explained and organized is as important as analyzing this knowledge from a mathematics discipline point of view. These *two aspects* are not isolated from each other in the analysis. I regard CK as pure mathematical content from a subject's discipline and PCK as multiple ways to represent algebra knowledge and the particular way to organize it and construct relevant mathematics exercises in order to make it accessible to learners. The word "content" is the central term in my study and it embodies these two aspects. Thus, my theoretical framework is CK-PCK. PCK tool in my framework makes it possible to find the following embedded *aspects* of PCK in the textbooks:

1. How the analyzed mathematics textbooks organize and represent algebra content related to solving quadratic equations in order to make this content accessible and comprehensible. This includes: examining the explanation and illustrations of algebra theory, concepts, rules and used examples as well as the application of algebra; finding connections within the algebraic content; discovering implied problems with a certain way of presenting algebra content which may cause students difficulties in learning algebra. In the application of this tool, the analysis focuses on looking for the embedded teaching trajectories in the textbooks.

2. What kind of mathematics exercises in the textbooks which are provided for learners to practice the related algebra content and facilitate learning in the embedded teaching trajectories; how they are constructed and what pedagogical aims they have. This includes

uncovering pedagogical aims of provided mathematics exercises, activities, and problems in the textbooks.

2.3 CK-PCK in mathematics textbooks

This study is about analyzing algebra content as CK and pedagogical content knowledge as PCK embedded in mathematics textbooks. The textbooks are studied as tertiary artifacts (Wartofsky, 1979) as mentioned above. How does a mathematics textbook relate to pedagogy then? Stray (1994) implicitly points out the relation between textbooks and pedagogy as below:

> … textbooks are the bearers of messages that are multiply-coded. In them the coded meanings of a field of knowledge (what is to be taught) are combined with those of pedagogy (how anything is to be taught and learned). […] textbooks can be conceived as a focal element in processes of cultural transmission (p. 1).

Stray (1994) has emphasized two important characters of textbooks: textbooks reflect subject matter content knowledge and pedagogical content knowledge related to a specific subject. Textbooks can function as artefacts when they are used by teachers and students. They embody and transfer the human knowledge according to theory of artefacts (Wartofsky, 1979; Säljö, 2007). Mathematics knowledge is presented in the form of written texts and mathematical representations, which constitute units of analysis in my study. As educational material, textbooks belong to textual resource materials produced for classroom use and can be regarded as pedagogic objects (Love & Pimm, 1996). Mathematics texts in mathematics textbooks or other forms of materials carry out two important pedagogical functions:

1. Creating a logical, mathematical progression, *from* past mathematical knowledge and experiences and *towards* preparing for the future content in the mathematics curriculum.
2. Embodying the development of cognitive structures in the learner from the conceptual aspect (Van Dormolen, 1986 cited in Love & Pimm, 1996).

Love and Pimm (1996) claimed that the mathematics texts are often logically structured in a linear form to evoke the sequential learning. They mentioned that mathematics textbooks are written to teach mathematics to learners. Textbook authors often regard the student as the main reader, and so they write the textbook from the teacher's position (Kang & Kilpatrick, 1992). There seems to be "a ghostly presence of the teacher" in texts for students (Love & Pimm, 1996, p. 385).

As Shulman (1986b) argued that PCK links both subject content and pedagogy, Love and Pimm (1996) found that textbooks have pedagogical functions when they present knowledge of subjects. A textbook is regarded as curriculum material in teaching and learning a subject. Thus in my study, a mathematics textbook does not only contain the knowledge of algebra as subject matter knowledge but also specific pedagogical content knowledge built into it by authors to formulate and represent algebra with most powerful illustrations, representations and examples as well as exercises in order to make algebra comprehensible and learnable. For example, distributive law is represented by a rectangle consisting of different small rectangles in order to visualize an abstract rule. Using geometrical representations is part of the embedded PCK in the textbook.

Geometrical representations like different combinations of rectangles and squares applied in the textbook are artifacts (Wartofsky, 1979) that embody algebra history and the fact that algebra often originated from geometrical ideas. They represent algebraic rules of operation and abstract representations such as quadratic equations and quadratic formula. Through these illustrations the algebra knowledge in the textbook has its connection with algebra history.

2.4 Other theoretical terminologies

Since the content analysis in this study involves both elementary algebra, abstract algebra and application of algebra, I here present a short background related to pure and applied mathematics by referring to De Lange (1996). De Lange pointed out that the dichotomy between pure and applied mathematics existed already in Euclid's *Elements*. The goal of mathematics was to study nature. In ancient Greek, geometric principles were embodied in the entire structure of the universe. Mathematics was recognized to embody the physical elements in the real world. In the 20th century, pure mathematics was created as a result of the expansion of mathematics and science. The idea that mathematics was not only a body of truths about nature, made mathematicians move their attentions to abstract mathematics isolated from problems of the real world. Abstraction, generalization and specialization are the three types of activity undertaken by pure mathematicians. Pure mathematics was regarded as good while applied mathematics was bad at that time. But, in recent decades, this attitude has changed. Applied mathematics received positive recognition with the development of information technology. Many social scientific fields saw mathematics as a useful tool. Social needs and technological requirements developed mathematical knowledge in society. This applied point of view appeared also in school mathematics in order to motivate students' interests in mathematics (De Lange, 1996). Teaching mathematics modeling relates to application of mathematical models (Lingefjärd, 2000). De Lange (1996) reported that after the 1980s, modeling and applications on one hand and problem solving on the other were merged together. People became convinced that students would benefit from applications and usefulness of mathematics. In the 1990s, the applied mathematics and pure mathematics were still an issue for discussions in the mathematics society. Based on an applied mathematics point of view, mathematics teaching linked the applications' real world to the students' own real world aiming at integrating mathematics learning with real world concepts. The learning process for developing mathematical concepts was assumed to start from the real world or concrete experience, to proceed to abstract conceptualization through reflective observation and active experimentation. This process was called conceptual *mathematization* (De Lange, 1996).

It was Freudenthal who grounded what came to me the theoretical frame work of RME: *realistic mathematics education* (De Lange, 1996). In the RME, the learning process starts with exploration of real appearances of mathematical concepts and structures, using reality as a source for *mathematization*. The characteristic of mathematization in RME is that it provides students with real world activities in which mathematics is explored during the "doing" process. The doing-activity process involves: first identifying the specific mathematics in a general context aiming at transferring the problem to a mathematically stated problem; then trying to discover regularities and relations through schematizing and visualizing the problem in different ways; when the problem is transferred into a mathematical problem, the problem is dealt with using mathematical tools which means that mathematical models are constructed; reflecting and refining mathematical models; finally generalizing the mathematical models in a more abstract conceptual way. Therefore, mathematization in RME is a synonym of modeling. Mathematics learning occurs through students solving real world problems. Teaching has to be reflexive and adapted, and organizing and facilitating students are the focus (De Lange, 1996).

In my study, algebra content analysis includes analyzing application of algebra that relates to the concept of *modeling*. Modeling refers to the construction of models or meaningful structures within one or more representational systems (Goldin, 2008). Aspects of models and modeling are used in RME (Van Den Heuvel-Panhuizen, 2003). In RME, Van Den Heuvel-Panhuizen (2003) claims that models play the role of bridging the gap between the informal understanding connected to the real and imagined reality on one hand and the understanding of formal systems on the other hand. In order to support learning processes, models have to be related to realistic, imaginable contexts and at the same time can be applied on a more advanced and general level (Van Den Heuvel-Panhuizen, 2003). For RME, mathematics occurs when students develop effective ways to solve problems (De Lange, 1996).

A model perspective is related to applications and usefulness of mathematics in mathematics education, in contrast with a pure mathematics perspective (De Lange, 1996). The concept of models used in the analysis of my study does not have quite the same sense as Realistic Mathematics Education models (Van Den Heuvel-Panhuizen, 2003) since there is neither any connection with realistic mathematics nor a relation with students directly, but they are related to algebra history and applied as pedagogical models. For example, the used rectangles and squares in the analyzed textbook are not only the geometrical figures for expressing areas, which originated from algebra history, but also models for representing the distributive law and the square rule. They have a common function of offering readers, including both teachers and students, visual illustrations that make sense of learning the distributive law and the completing square method as well as quadratic expressions.

2.5 Summary

To sum up, in this study, I analyze the mathematical content related to algebra in particular quadratic expressions and equations presented in the mathematics textbooks as tertiary artifacts (Wartofsky, 1979). The analytical framework applied for this study is CK-PCK (Mishra & Koehler, 2006; Shulman, 1986b). Analyzing the algebra content presented in the textbook does not only focus on examining the content from the point of view of mathematics as a discipline concerning the subject matter content knowledge, it also looks for embedded PCK regarding content organization and teaching trajectories. In many places of this thesis, I use expressions *embedded teaching trajectories* and teaching progression. The word “teaching” in these expressions refers to teaching in a hypothetical meaning, which is possible and imagined rather than how it unfolds in classroom interaction.

I hope that my study will be useful for understanding of teaching quadratic equations from a PCK point of view. I examine the content of quadratic equations and seek the embedded teaching trajectories with the goal of teaching quadratic equations. I ask the questions: What is presented in the textbook? Why is the content in the textbook organized as it is?

3. Related Mathematical Content

The core content analyzed in this study contains elementary algebra related to quadratic equations; such as quadratic polynomials, multiplication of two binomials, distributive property, factorization, completing the square, and quadratic formula. Factorization is discussed in relation to both elementary and advanced algebra. A literature review related to these topics is carried out and presented in this chapter. The review study finds that the algebra content in the textbooks has strong connections with the history of algebra. In this chapter, the core content will be presented in three parts: 3.1 The history of algebra 3.2 Different approaches to solving quadratic equations, and 3.3 Factorization in abstract algebra.

3.1 A review of algebra history and its development

This part of chapter three aims at introducing algebra history in relation to solving quadratic equations. Algebra has a long history in mathematics development. Quadratic equations and polynomials belong to the area of algebra in mathematics. What is algebra? Colin Maclaurin in his 1748 algebra text defined it like this:

> "Algebra is a general method of computation by certain signs and symbols which have been contrived for this purpose, and found convenient. It is called a universal arithmetic, and proceeds by operations and rules similar to those in common arithmetic, founded upon the same principles." (Katz & Barton, 2007, p. 185).

Leonhard Euler, in his own algebra text in 1770, defined algebra as "the science which teaches how to determine unknowns' quantities by means of those that are known." (ibid., p. 185).

Katz and Barton (2007) categorize the historical development of algebra in four stages: the rhetorical stage, the syncopated stage, the symbolic stage and the purely abstract stage, but they also name another four conceptual stages:

> "the geometric stage, where most of the concepts of algebra are geometric; the static equation-solving stage, where the goal is to find numbers satisfying certain relationship; the dynamic function stage, where motion seems to be an underlying idea; and finally the abstract stage, where structure is the goal" (p. 186).

As an old science, algebra has a complicated historical background according to Katz and Barton (2007). Algebraic procedures have developed slowly. There are different opinions about where the evolution of the term "algebra" started. It is commonly believed that algebra first appeared among the Egyptians, the Babylonians, the Greeks or the Arabs. The geometrical influence on algebraic reasoning was strong in ancient Greece. However, the word algebra originated in Baghdad, where the Arabic scientist al-Khwarizmi (A.D. 780-850) published a short book about calculating with the help of al-jabr and al-muqabala[1]. Today's algebra has its root in Arabic algebra. Western mathematics tended to turn algebraic

[1] *Al-jabr* means restoration and *al-muqabla* means reduction.

operations into symbols and later developed abstract algebra. The process of algebra development was slow and the whole history lasted 4000 years (Katz & Barton, 2007).

The rhetorical stage originated from geometry ideas

In his book *Unknown Quantity*, Derbyshire (2006) points out that algebra began very early in recorded history. The first algebra texts are dated to the first half of the second millennium BCE, from 37 or 38 centuries ago, and were written by people living in Mesopotamia and Egypt. During the Hammurabi period from about 1790 to 1600 BCE, the Babylonians started their civilization by pressing written words in patterns called cuneiform or wedge-shaped stylus into wet clay. Many tablets in cuneiform had mathematical algebraic content. Their mathematical texts were of two kinds, table texts and problem texts. The table texts were lists of multiplication tables, tables of squares and cubes as well as advanced lists like the famous Plimpton 322 tablet, which is about Pythagorean triples. At that time, the Babylonians had neither defined zero nor negative numbers.

The Babylonians of Hammurabi’s time had no proper algebraic symbolism. All mathematical problems were expressed in words, for example unknown quantity in Sumerian’s Akkadian text, was expressed as *igum* (length) and *igibum* (width) as reciprocal (Derbyshire, 2006). The application of algebra might have had its reason in the need of measuring land areas. At the rhetorical stage, all mathematical statements and arguments were made in words and sentences (Derbyshire, 2006). Babylonian mathematics has two roots, one is accountancy problems and the other one is a “cut and paste” geometry (Katz & Barton, 2007, p. 191), probably developed for understanding the division of land. Many ancient Babylonian clay tablets contain quadratic problems of which the goal was to find such geometric quantities as the length and width of a rectangle. As an example, a clay tablet text tells that the sum of the length and width of a rectangle is 6½, and the area of the rectangle is 7½ (Derbyshire, 2006; Katz & Barton, 2007). What are the length and the width of this rectangle? The tablet describes in detail the steps the writer went through.

First, the writer halves 6½ to get 3¼. Next, he squares 3¼ to get $10\frac{9}{16}$. From this area, he subtracts the given area 7½, leaving $3\ \frac{1}{16}$. The square root of this number is extracted: 1¾. Finally, the length is 3¼ + 1 ¾ = 5, while the width is 3¼ − 1¾ = 1½ (Katz & Barton, 2007). The whole process can be translated into part of a quadratic formula, shown in *a* below. Since the Babylonians did not know about negative numbers, the only solution for them was positive, hence their algorithm did not deliver the two solutions to the quadratic equation, so their formula is slightly different from the modern quadratic formula, as shown in *b* below (Derbyshire, 2006):

a) $1\frac{3}{4} = \sqrt{\left(6\frac{1}{2} \div 2\right)^2 - 7\frac{1}{2}}$ b) $x = \sqrt{\left(6\frac{1}{2} \div 2\right)^2 - 7\frac{1}{2}} \pm \left(6\frac{1}{2} \div 2\right)$

If we denote the sum of the length and the width of the rectangle as *b* and the given area as *c*, this formula will be as shown in *c*, which is the modern quadratic formula.

c) $x = \sqrt{\left(\frac{b}{2}\right)^2 - c} \pm \left(\frac{b}{2}\right).$

Even though there are different interpretations of Neugebauer and Saches’ translation of the Babylonian’s tablets for this text on finding the length and width of a rectangle (Katz &

Barton, 2007), it is clear that the text from the tablets is dealing with a geometric procedure. The problem was solved in words but with geometric ideas. This was the beginning of algebra.

According to Katz and Barton (2007), the Greek mathematician Euclid (300 B.C.) in his Book II of *Elements* solved some algebraic problems by manipulating geometric figures, but based on clearly stated axioms. The geometrical method is more explicit in Euclid's work *Data* than in *Elements*. The following example illustrates how Euclid solved a quadratic equation using a geometrical method. Euclid defined "proposition 1" which is like axiom 1: "If two straight lines contain a given area in a given angle, and if the sum of them be given, then shall each of them be given (i.e., determined)" (Katz & Barton, 2007, p. 189). Euclid set up a rectangle ACFS with the two sides $x = AS$ and $y = AC$. Then a line was drawn from point S to a point B so that $BS = AC$ and the completed rectangle was *ACDB* (Figure 1). Suppose that $AB = x + y = b$ was given and the area of rectangle *ACFS* was given denoted as c, what were the two sides x and y of the rectangle ACFS?

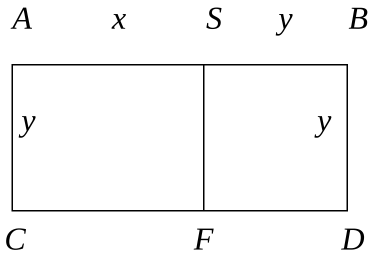


Figure 1. The first step in using geometrical figures to solve a quadratic equation according to Euclid (Katz & Barton, 2007, p. 189).

In order to find the length and the width of the rectangle, Euclid bisected *AB* at *E*, constructed the square on *BE*, then claimed that this square was equal to the sum of the rectangle *ACFS* and the little shadow square at the bottom (Figure 2).

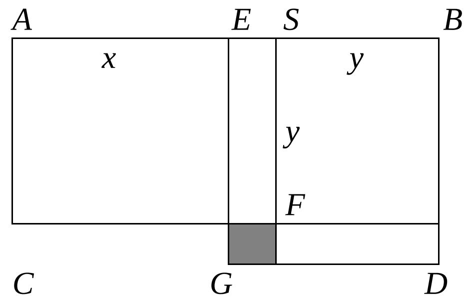


Figure 2. The second step in using geometrical figures to solve a quadratic equation according to Euclid (Katz & Barton, 2007, p. 189).

According to Euclid, the area of the rectangle *ACFS* was given, which was c, and the area of the new square *EGDB* was also given which was $(b/2)^2$ since:

$$EB = \frac{AS + SB}{2} = \frac{x + y}{2} = \frac{b}{2}.$$

The equivalent relationship between the areas can be formulated as a quadratic equation:

$$\left(\frac{b}{2}\right)^2 = c + \left(x - \frac{b}{2}\right)^2 \text{ or } \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2} - y\right)^2$$

The critical step in solving the problem is that Euclid found this equivalent relationship geometrically and made use of this relationship to find the solutions of the problem, so the length and width of the rectangle *ACFS* are:

$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c} \text{ and } y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

These two formulas are almost identical with the Babylonians' solutions in rhetoric expressions. The difference is that Greek algebra was based on geometric manipulation while Babylonian algebra was based on rhetoric manipulation with geometrical ideas. In general, the early stage of algebra from ancient Babylon and Egypt to Greek was mainly geometrical.

The syncopated stage–the beginning of the static equation-solving stage

Derbyshire (2006) presents that in Roman Egypt, probably in the second or third century CE, algebra was at the syncopated stage which means written algebraic texts were expressed in words but involved in special symbols-abbreviations. According to the history of recorded mathematics, one of the pioneers in using these special symbols to solve equations with only numbers but no connection with geometry was Diophantus who lived in Alexandria in Egypt around the third century. Diophantus used the Greek alphabet for writing numbers. He wrote a treatise titled *Arithmetica*, of which less than half has been maintained today. The surviving part of his work consists of 189 problems in which the object was to find numbers, or families of numbers, satisfying certain conditions. In mathematics today, Diophantus' mathematical analysis is known as number theory (Smith, 2006). He wrote the coefficient after the variable, instead of before it as we do. He used the Greek letter ς for an unknown quantity, in modern algebra written as *x*. Most of his book dealt with indeterminate equations which contained more than one unknown and a potentially infinite number of solutions. His problem was that he could not represent more than one unknown; instead he solved quadratic equations with two unknowns through substituting one by another. Diophantus did not use negative numbers but used elementary principles of expansion, factorization, gathering up of like terms and simplification. Diophantus created his own literal symbolism with the use of special letter symbols for the unknown and its powers, for subtraction and equality (Derbyshire, 2006).

From Diophantus, algebra history moved into another conceptual stage, the equation-solving stage according to Katz and Barton (2007). In India, quadratic formula appeared without any geometric support. Brahmaggupta (598-665) was one of the first mathematicians who could systematically handle negative numbers and zero. He gave a general solution to quadratic equations and realized that there were two roots for a quadratic equation. It was possible that one of the roots was a negative number. Baskharacharya (1114-1185) solved mathematics problems with quadratic equations in his book *Siddhanta Siromani* (Mathematical Pearls). They presented an algorithm to reduce a quadratic equation to a first degree equation (Olteanu, 2007).

It is commonly believed that the first true algebra text was the work on *al-jabr* and *al-muqabala* by Mohanmmad ibn Musa al-Khwarizmi (780-850), written in Baghdad around 825 (Katz & Barton, 2007). The word algebra came from the title of this work. The word *al-jabr* means restoration or reestablishment that is to eliminate negative terms through adding the same terms to both sides of equations. The word *al-muqabalas* means balance, which is to divide every term in a quadratic equation by the coefficient of the second degree's term (Josephs, 1991 in Olteanu, 2007). The first part of his book is a manual for solving linear and quadratic equations. Al-Khwarizimi classified equations into six types, three of which were mixed quadratic equations. For each type, he presented an algorithm for its solution. Five of the six types of equations were quadratic equations which can be expressed in modern forms, $ax^2 = bx; ax^2 = c; ax^2 + bx = c; ax^2 + c = bx; ax^2 = bx + c$. Here is an example of how Al-Kwarizimi would solve the equation $x^2 + 10x = 39$:

> Take the half of the number of the things, that is five, and multiply it by itself, you obtain twenty-five. Add this to thirty-nine, you get sixty-four. Take the square root, or eight, and subtract from it one half of the number of things, which is five. The result, three, is the thing. (Kvasz, 2006, p. 292)

Like Babylonian mathematicians, al-Khwarizimi's algorithm is entirely verbal. The geometrical explanations of al-Khwarizimi's algorithm can be translated into today's "square completing method" (Olteanu, 2007). Using the example of the solving quadratic equation $x^2 + 10x = 39$, the completed geometrical procedures are illustrated below in figures 3, 4, and 5 (Olteanu 2007, p. 30).

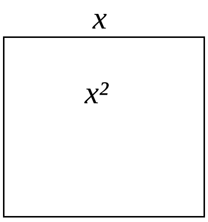


Figure 3. A square used by al-Khwarizmi for solving quadratic equations

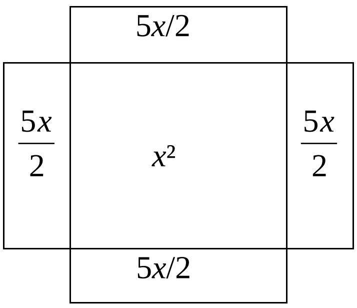


Figure 4. The second step for completing a square according to al-Khwarizmi

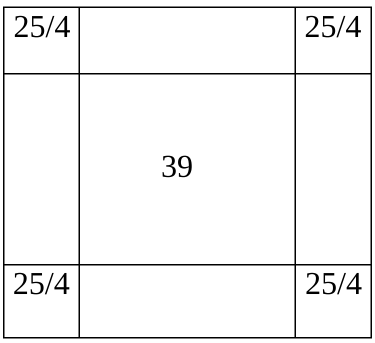


Figure 5. The third step for completing a square according to al-Khwarizmi

According to Olteanu (2007), al-Khwarizimi started with a square whose side is x and area is x^2 (see Figure 3). Then he added four equal rectangles whose areas in total were $10x$ along each side of the square, that is $10x = 4 \cdot (5/2) \cdot x$. Each rectangle's area is thus $(5/2) \cdot x$ with its length x and its width 5/2 (see Figure 4). The sum of the big square and four rectangles was given, that was 39. The equivalence relationship was: $x^2 + 10x = 39$. Finally, Figure 4 was completed by adding four small equal squares which had an area of the size $(5/2) \cdot (5/2) = 25/4$ for each small square and the sum of them was 25. Through adding this sum to both sides of the equation, the area of the biggest square in Figure 5 obtained was 64. Written as an equation:

$$x^2 + 10x + 4 \cdot \frac{25}{4} = 39 + 4 \cdot \frac{25}{4}$$

The side of the biggest square was 8 and had its relation with other sides of different squares expressed in the first degree equation, $8 = (5/2) + x + (5/2)$, and then x was 3. With "cut-and-paste" geometry (Katz & Barton, 2006, p. 191), al-Khwarizimi reduced the second degree equation to a first degree equation and thereafter solved it.

The symbolic stage

At this stage of algebra, "all numbers, operations, relationships are expressed through a set of easily recognized symbols, and manipulations on the symbols take place according to well-understood rules" (Katz & Barton 2007, p. 186). The ancient algebra and geometry had developed simultaneously in Egypt, Persia, Greece, India and China. Following Medieval Islamic scholars, who gave us the word "algebra," Western Europe began the struggle for the development of algebra starting with some algebraists from Italy.

Derbyshire (2006) states that Italian mathematician Leonardo Pisano, later known as Fibonacci, in the 12th and 13th centuries had traveled in Persia, India and China. When he returned to Italy, he brought back wider knowledge of arithmetic and algebra. His book *Liber abbaci* was the best math textbook since the end of the Ancient World. His book was credited with having introduced Indian numerals, including zero, to the West, and his algebraic skills were shown in two other books after this one. With the arrival of printed books during the second half of the 15th century, the development of algebra speeded up. Several Italian mathematicians, including Girolamo Cardano, had figured out how to solve cubic and quadratic equations. Algebra became purely abstract with some exceptions, for example the English mathematician Robert Recorde who lived in the 16th century and created quadratic problems from real world experiences (Derbyshire, 2006).

It was in France that algebra developed into a well-organized literal symbolism. In his work *In artem analyticem isagoge*, French mathematician François Viète (1540-1603), was the first mathematician to use letters representing numbers systematically and effectively in the late 16th century (Derbyshire, 2006). He made a range of letters available for many different quantities. This was the beginning of modern literal symbolism. Viète's unknown quantity was divided into two classes. Unknown quantities, which means "things sought," were denoted by vowels like *A, E, I, O, U,* and *Y*; while "things given" were denoted by constants like *B, C, D* etc. For example, his *A* is our unknown *x*. Viète was a pioneer in the study of equations. His two papers on the theory of equations were published twelve years after his death. In the second paper, titled *De equationem emendatione* (On the perfecting of equations), Viète opened up the line of inquiry that led to the study of the symmetries of an equation's solutions to Galois theory, the theory of groups, and of all modern algebra. He found the relationship between the solutions of the equation and the coefficients for the first five degrees of equations in a single unknown. To explain this in our modern symbols, we suppose that the two solutions of the quadratic equation $x^2 + px + q = 0$ are α and β which means that $x_1 = \alpha$; $x_2 = \beta$. Since only α and β, and no other values of x, make this equation true, the following must be true: $(x - \alpha) \cdot (x - \beta)$. This is just a rewritten form of the same equation. If we multiply out those parentheses, this rewritten equation turns to be: $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Compared to the original equation, the relationships between the solutions and the coefficients lead to the conclusion that $\alpha + \beta = -p$; $\alpha\beta = q$ (Derbyshire, 2006). It is said that Viète discovered the solution formula today called "quadratic formula for general quadratic equations," which is:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for a general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

Another French mathematician and philosopher who had strong influence in the history of algebra was René Descartes (1596-1650). His idea to use the Cartesian system of coordinates, which was named after the Latin form of his name, has developed both algebra and geometry. In his work *La géométrie* (1637), by using numbers to identify points in Cartesian coordinates, Descartes connected geometrical objects to algebraic numbers and made the classical geometry become analytical geometry. He took up the plus and minus signs from the German Cossists of the previous century and also the square root sign. From Descartes' time, the symbol of an unknown has become represented by x (Derbyshire, 2006).

Descartes developed the idea of functions in Cartesian coordinates though ideas about function can be traced back to the Islamic mathematician Sharaf al-Din al-Tusi from Persia (Katz & Barton, 2007) and Klaudius Ptolemaios about 2,000 years ago (Olteanu, 2007). Descartes declared that every curve in a Cartesian coordinate system has an equivalent equation which can represent the points on the curve or vice versa and every equation containing x and y can be represented by a curve through its coordinate points. However, the word function was introduced by Gottfried Wilhelm Leibniz in 1693 and the definition of function was stated by Leonhard Euler in his work *Introductio analysin infinitorium* in 1748. Euler had also introduced the symbol $f(x)$ denoting f as a function relying on a variable x (Olteanu, 2007). In the 18th century, since the discovery of the calculus by Newton and Leibniz, algebra went into an area of analysis: the study of limits, infinite sequences and series, functions, derivatives, and integrals (Derbyshire, 2006).

The purely abstract stage – algebraic structure

Katz and Barton (2007) point out that since the 17th century, algebra has not solely been about finding solutions for different degrees of equations anymore, instead mathematicians started to integrate algebra with astronomy and physics. Johann Kepler and Galileo Galilei were interested in curves and finding a mechanism for representing motion instead of quantity with numbers. However, their arguments are not algebraic symbolic but geometric. It was Fermat and Descartes, now regarded as the fathers of analytic geometry, who showed how to represent a curve described verbally with numbers and algebraic symbols. Analytic geometry thus became a mechanism for representing motion. Newton, in his *Principia*, picked up on Fermat and Descartes' representation and developed the calculus. With the invention of the calculus, mathematical problems were solved by curves, not just points. Algebra grew more and more to present paths of motion. In order to be able to judge if the algebraic manipulations were correct, axioms were formulated for arithmetic applied for algebraic manipulations. Late in the 18th century, Lagrange introduced the idea of permutations into the search for solutions. Thereafter, Galois developed methods for determining under what conditions polynomial equations are solvable. New abstract algebraic concepts were found like "field" and "group" during the early 19th century. In 1854, Ceyley gave an axiomatic definition of a group. During the 1890s, this definition entered textbooks along with the axiomatic definition of a field, an idea which had roots in the work by Galois. At the beginning of the 20th century, algebra became less about finding solutions to equations and more about looking for common structures in many mathematical objects defined by sets of axioms (Katz & Barton, 2007).

Summary on the review of algebra history

This review has focused on some essential moments related directly to the algebra content presented in the analyzed textbook in the present study:

- Solving quadratic equations by completing-the-square approach originated from a geometrical cut-and-paste approach used by ancient Babylonians whose geometrical ideas were written on clay tablets.
- Quadratic formula had its origin in ancient Babylon but was developed by Euclid with a geometrical method. The geometrical figures based on his method represent the procedures of constructing quadratic formula.
- Relations between roots of a quadratic equation and the coefficients and a constant of the equation were discovered by the French mathematician Viète. These relations can be described as $\alpha + \beta = -p$; $\alpha\beta = q$ if α and β are two roots of a quadratic equation $x^2 + px + q = 0$.
- In the 17th to 18th centuries, the concept and definition of function was created
- During 19th and the beginning of 20th century, algebra became more and more abstract and algebra structure was the focus.

This review has recognized algebra content presented in the textbook related to the history of algebra, in particular the use of geometrical ideas and methods of solving quadratic equations. This geometrical approach is the origin of the completing the square method. The quadratic formula has been found to relate to Euclid's geometrical method. The relations between roots and the coefficients and constant of an equation were discovered by Viète.

3.2 Three approaches to solving quadratic equations

In the previous part of chapter three, I have reviewed the rhetorical and geometrical approaches of solving quadratic equations in the history of algebra. In order to get more insights into quadratic equations' solving approaches–such as quadratic formula, completing square, and factorization–I have carried out another literature review study in the area of factorization and different approaches of solving quadratic equations. This part presents three common methods for solving quadratic equations discussed in secondary mathematics educational fields in some previous studies (Allaire & Bradley, 2001; Bossé & Nandakumar, 2005; Hoffman, 1976; Leong et al., 2010; Kemp, 2010; Kennedy & et al., 1991; Nataraj & Thomas, 2006; Vaiyavutjamai & Clements, 2006; Vinogradova, 2007; Zhu & Simon, 1987): 1) *Completing the square* 2) *Factorization* 3) *The quadratic formula.* The reason to present these three methods is that they are not only common topics for discussion in the articles searched in this area concerning mathematics education at upper secondary school but also presented in the analyzed mathematics textbook in my study.

1) Completing the square

Some mathematics education articles advise mathematics teachers to simplify ancient-time mathematician al-Khwarizmi's method of completing the square based on geometrical ideas and then present the simplified version to students (Allaire & Bradley, 2001; Vinogradova, 2007). By doing so, the teachers relate mathematics quantity to physical objects in a visual way. Didactically, teachers can start with a concrete example in which students can understand the content visually. Later, from the concrete example, the teachers lead the students to another example expressed in algebraic symbols. As an example: study a rectangle whose area is $x(x + 10)$ and suppose this area is 39, that is $x(x + 10) = 39$. Four steps are followed to build a new square, which is actually called completing the square.

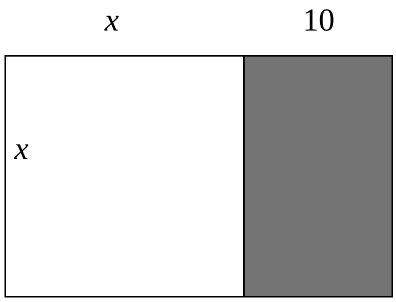


Figure 6. A rectangle representing a quadratic equation

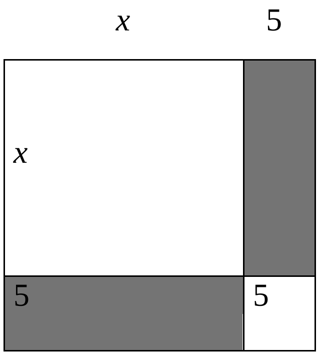


Figure 7. The procedures of completing the square for solving quadratic equations using a geometrical approach:

A. Begin with a rectangle of the area $x(x + 10)$, that is with the short side as x and the long side $(x + 10)$. The shaded area is $10x$ (Figure 6**)**.
B. Divide the rectangle $10x$ into two small equal rectangles with the size $5x$ each and move them to each adjacent side of the x^2 square (Figure 7). The total area including the big blank area and the area of two shaded rectangle is still $39 = x(x + 10)$.
C. In Figure 7, a new square is shaped in the lower-right hand corner with the side 5 and area 25. By adding this small square to the geometrical figure, the large square with an area of $x^2 + 2(5x) + 25$ is "completed." Therefore, we can write $39 + 25 = x(x + 10) + 25$ or $x^2 + 2(5x) + 25 = 64$. The area of the large square is now $(x+5)^2$ since the side has a length of $x + 5$. Therefore $(x + 5)^2 = 64 \Rightarrow x + 5 = 8 \Rightarrow x = 3$.

After these three steps, a solution to this quadratic equation is derived by completing the square geometrically (Allaire & Bradley, 2001). This geometrical approach has actually its origin in al-Khwarizimi's algorithm for solving quadratic equations using geometrical ideas which was presented in the first part of this chapter. The didactic purpose of using the geometrical figures here is to visually offer opportunities for the student to understand the process of approaching the method of completing the square.

2a) Factorization of quadratic expressions

Factorization has been discussed as an alternative way to solve quadratic equations (Vaiyavutjamai & Clements, 2006). This is often related to teaching factorizing quadratic expressions first. Factorizing quadratic expressions is a common didactic topic at secondary level in mathematics education (Bossé & Nandakumar, 2005; Hoffman, 1976; Leong et al., 2010; Kemp, 2010; Kennedy & et al., 1991; Nataraj & Thomas, 2006; Zhu & Simon, 1987).

Hoffman (1976) presents two approaches (A and B) for factorizing quadratic expressions through observing and grouping. By taking the second degree polynomial $3x^2 + 7x + 2$ as an example, the following steps are carried out to factorize this polynomial.

A. Decomposition of the linear term.

1) Multiply $3x^2$ and 2 to get $6x^2$.
2) Decompose $7x$ into the sum of two terms whose product is $6x^2$.
   This gives $7x = 6x + x$. Factorize $3x^2 + (6x + x) + 2$:
   $3x^2 + (6x + x) + 2$
   $= (3x^2 + 6x) + (x + 2)$
   $= 3x(x + 2) + 1(x + 2)$
   $= (x + 2)(3x + 1)$

The approach is based on the observation of distributive law (a) and the commutative and associative laws (b):

a) $(ax+b)(cx+d) = acx^2 + adx + bcx + bd = acx^2 + (ad+bc)x + bd$

b) $(acx^2)(bd) = acbdx^2 = (adx)(bcx)$.

In this approach, observing the relationships between coefficients to x^2 and x as well as the constant is the important starting point. It is often called the “guess-and-check” method (Kemp, 2010).

B. Making the coefficient of the quadratic unknown become 1 (i.e. $a = 1$ in $ax^2 + bx + c$).
In this case (Hoffman, 1976):

$$3x^2 + 7x + 2 = \frac{1}{3}\left[(3x)^2 + 7(3x) + 6\right] = \frac{1}{3}\left[(3x+6)(3x+1)\right] = (x+2)(3x+1).$$

It may be confusing to some students why the polynomial has to multiply with 3 and is divided by 3 at the same time which means $3 \cdot (1/3) = 1$. The operational procedures are associated with distributive and associative laws of integers.

These two approaches of factorization are dependent on skills in multiplication and division which expose students’ knowledge of arithmetic and the number theory. Factoring quadratic expressions is the first step to solve a quadratic equation. Solving the quadratic equation $3x^2 + 7x + 2 = 0$ by factorization is actually to decompose the polynomial into a factoring form $(x+2)(3x+1) = 0$ which is equivalent to the polynomial form. The essential step to solve this equation is to follow the null-factor law[2] (Gennow, Gustafsson, & Silborn, 2005b, p. 106),

[2] *nollprodukt* in Swedish

that is: for real numbers p and q; $p \cdot q = 0$ if and only if $p = 0$ or $q = 0$. It means that either of the binomials on the left side of the equation has to be equal to zero in order to satisfy the equivalent relation of this quadratic equation. The solutions of two roots $x_1 = -2$ and $x_2 = -(1/3)$ are obtained through making either $x + 2 = 0$ or $3x + 1 = 0$.
These two approaches are not the only ways to factorize quadratic expressions. There are many methods discussed in the field of mathematics didactics. The three following studies are being presented to show the different methods for doing factorization.

Study one: Teaching factorizing quadratics is influenced by the different cultures where it is taught (Kemp, 2010). In the same classroom, different cultures contribute to the richness of using different methods for factorizing quadratic expressions. According to Kemp (2010), Russian students use the standard quadratic formula to factorize quadratic polynomials:

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

Students from China and Hong Kong use the cross multiplication method based on the guess-and-check process to work with factorization. For example, factorizing the quadratic expression $6x^2 + 7x + 2$ by looking at the factors of the coefficient of x^2 which can be factorized as factors 6 and 1 or 3 and 2; and factors of the constant which can be 2 and 1 or -2 and -1. Then the process of finding the value of the coefficient to x, in this case 7, is carried out by cross-multiplying these factors written in a matrix and adding the two multiplications until finding the value 7, as shown in figures 8 and 9 (Kemp, 2010, p. 44).

$6x$	2	1	-1	-2
x	1	2	-2	-1
	$8x$	$13x$	$-13x$	$-8x$
	$= 2x + 6x$	$= x + 12x$	$= -x + -12x$	$= -2x + -6x$

Figure 8. The cross multiplication method for factorization quadratics by Chinese students

$3x$	2	1	-1	-2
$2x$	1	2	-2	-1
	$= 7x$			

Figure 9. The value of the coefficient to x is found after a few steps of trying.

The cross multiplication procedure is halted when the value 7 is reached, the factorized form is derived by reading along the rows: $(3x+2)(2x+1)$.

Study two: A Singaporean project was carried out as a lesson study about teaching factorizing quadratic expressions through the use of concrete algebra tiles (Leong et al., 2010). The cross multiplication method to factorize quadratic expressions is evidenced to be arbitrary and fails to make sense for students in their pre-test of the lesson study. In order to make a pedagogical change and improve students' learning of this topic, the lesson study team works out a teaching approach of using algebra tiles (Howden, 1985; Norton, 2007) to factorize quadratic expressions with small integers as coefficients and checking the factorization result by changing the used algebra tiles into an equivalent rectangle diagram (see Figure 10).

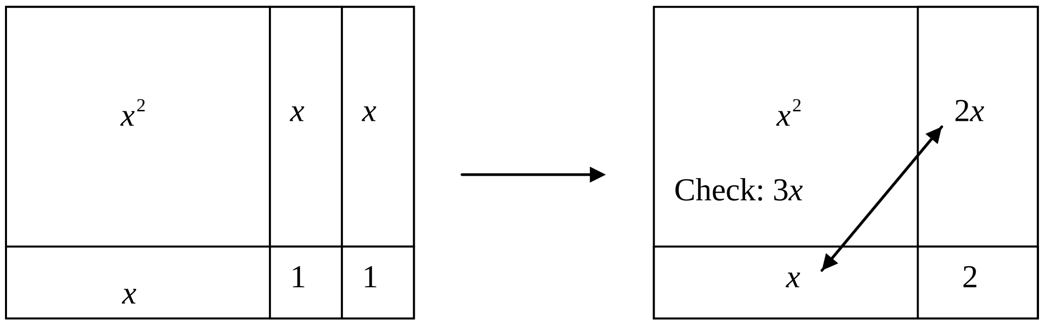


Figure 10. From algebra tiles to a rectangle diagram in factorization in the Singapore project (Leong et al., 2010, p. 22)

Figure 10 shows how a quadratic expression $x^2 + 3x + 2$ is factorized by using algebra tiles consisting of a bigger square x^2 and two small same squares in the value of 1 each as well as three same rectangles in the value of x each. The factors of the expression are found through adding a rectangle to a square vertically and adding a rectangle to the two small equal squares horizontally, that is $(x+1)(x+2)$. The three identical rectangles are the three units represented as coefficient 3 and the two identical squares are the two units represented as constant 2 in the used algebra tiles in Figure 10. In order to see the link between the two representations of factorizing $x^2 + 3x + 2$, a rectangle diagram equivalent to the used algebra tiles is derived (see Figure 10). In the rectangle diagram, the total area of the bigger rectangle is the sum of three different rectangles and a square. By cross-adding $2x$ and x, students can check the result of factorization. Factorizing quadratic expressions with negative coefficients can also be carried out by labeling the sides of a rectangle with negative numbers.

The post-test after the two lessons shows significant improvements and evidences of successful use of algebra tiles and rectangle diagram in factorizing quadratic expressions. The pedagogical purpose of this study is to help students understand manipulating factorization through a concrete and visual geometrical representation of factorization and improve their awareness of the link to symbolic algebra (Leong et al., 2010).

Study three: Jackman (2005) demonstrates a systematic procedure for factorizing second degree polynomials. A second degree polynomial can be written as:
$ax^2 + bx + c = (px + q)(rx + s)$ where a, b, c, p, q, r, s are all integers.
By multiplying the parentheses, the polynomial becomes:
$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$, so that $a = pr$, $b = ps + qr$, $c = qs$.
Writing the factors of a and c on two lines:

$$\begin{array}{ccc} p & & q \\ & \times & \\ r & & s \end{array}$$

Finding the right pairs of factors for a and c is done by listing the groups of products of different factors, for example in the polynomial $6x^2 + 5x - 4$, the factor pairs for 6 are (6, 1); (1, 6); (3, 2); (2, 3) and the factor pairs for (-4) are (4, -1); (-4, 1); (2, -2); (-2, 2); (1, -4); (-1, 4). Writing them on two lines creates the matrix:

$$\begin{array}{lllll} p & 6\ 1\ 3\ 2 & & q & 4\ {-4}\ 2\ {-2}\ 1\ {-1} \\ & & \times & & \\ r & 1\ 6\ 2\ 3 & & s & {-1}\ 1\ {-2}\ 2\ {-4}\ 4 \end{array}$$

In the next step, the values of ps and qr are calculated until the value of $b = 5$ is obtained.
…

$2 \cdot (-1) + 3 \cdot 4 = 10;$
$2 \cdot 1 + 3 \cdot (-4) = -10;$
$2 \cdot (-2) + 3 \cdot 2 = 2;$
$2 \cdot 2 + 3 \cdot (-2) = -2;$
$2 \cdot (-4) + 3 \cdot 1 = -5;$
$2 \cdot 4 + (-1) \cdot 3 = 5.$
…

After a tiresome procedure, two computing results obtain the same value of 5. They are: $3 \cdot (-1) + 4 \cdot 2 = 5$; $2 \cdot 4 + (-1) \cdot 3 = 5$. The polynomial can be factorized into $(3x+4)(2x-1)$ and $(2x-1)(3x+4)$ based on these two computing results which are the same. The roots are $x_1 = -(4/3)$; $x_2 = (1/2)$. The shortcoming of this method is that it is ineffective and complicated. The addition of matrixes is almost impossible and unbearable. This study describes the same procedure as the cross multiplication method described above in Study one (Kemp, 2010). However factorization can be carried out in different ways. Which method that is the more efficient one depends on the integers of the quadratic expression. Mathematics classrooms in different cultures teach different methods for factorization.

2b) Five different kinds of quadratic equations solved by factorization

Different kinds of quadratic equations have been generalized in a study of the factorability of factorization as an approach to solving quadratic equations (Bossé & Nandakumar, 2005). Based on those different kinds of quadratic equations, I will list five different kinds and solve them by factorization in order to make clear how different factorizations depend on various kinds of quadratic equations.

The coefficients a, b, p, q and constant c illustrated in the following quadratic equations are defined not to be equal to zero, that is $a \neq 0$ $b \neq 0$ $c \neq 0$….

Type 1. The first kind is the obviously factorable type (Bossé and Nandakumar, 2004): $ax^2 + bx = 0$. The equation is factorized as $x(ax + b) = 0$ where the roots are $x_1 = 0$; $x_2 = -(b/a)$.

Type 2. The second kind of quadratic equations is $(ax)^2 - b^2 = 0$. The factors of this second degree polynomial are $(ax)^2 - b^2 = (ax + b)(ax - b)$, so the roots to this type of quadratic equations are $x_{1,2} = \pm(a/b)$. This kind of factorization is based on using the difference-of-squares formula. For example, $9x^2 - 4 = 0$ can be solved by factorizing into $(3x + 2)(3x - 2)$, and then the roots are $x_1 = -(2/3)$; $x_2 = (2/3)$. This kind of factorization is also obviously factorable.

Type 3. The third kind of quadratic equations is $(ax)^2 \pm 2abx + b^2 = 0$. This type of equations can be solved directly by factoring into two identical binomials:
$(ax)^2 \pm 2abx + b^2 = (ax \pm b)^2 = (ax \pm b)(ax \pm b) = 0$.
The double roots obtained from this factorization are $x_{1,2} = \pm(b/a)$. This method utilizes square rules. For example, $4x^2 - 4x + 1 = 0$ can be solved by factorizing into two identical factors: $(2x-1)(2x-1) = 0$, the roots are $x_{1,2} = (1/2)$.

Type 4. The fourth kind of quadratic equations is $x^2 + px + q = 0$. In this equation, the coefficient of the x^2-term is 1 with p and q as positive or negative integers. Type 4 equations can always be solved by the approaches of completing the square and quadratic formula. However, they can be solved by factorization, too, since the quadratic expressions are factorable by using distributive laws reversely. The factorization methods presented in 2a of this chapter can be applied for this kind of quadratic equations. The roots are obtained through finding two factors for q, for which the sum is equal to $-p$. This way, the two factors become two roots of a quadratic equation. Denoting the two factors or two roots as r_1 and r_2, the relationship between coefficient and constant is: $q = r_1 \cdot r_2$; $p = -(r_1 + r_2)$. Quadratic equations can be written as the product of two first degree binomials or factoring form: $x^2 + px + q = x^2 - (r_1 + r_2)x + r_1 r_2 = (x - r_1)(x - r_2) = 0$.
Through the factoring form, the roots of this equation are obtained, that is $x_1 = r_1$; $x_2 = r_2$. The core of factorizing type 4 equations is seeking the connections between coefficients and roots by making use of basic arithmetic skills. It is often more effective in this case to use factorization than to use quadratic formula or completing the square if the constant q in the equations is easily factorized into two factors which satisfy relations such as $q = r_1 \cdot r$; $p = -(r_1+r_2)$. This can be shown with the example $x^2 - 12x + 35 = 0$. In this equation, $p = (-5) + (-7) = -12$, and $q = (-5)(-7) = 35$. By making use of this relation, the equation can be solved by the factorization method. Factorizing the left side of the equation obtains the product $(x - 5)(x - 7)$, the equation can be rewritten into factorized form $(x - 5)(x - 7) = 0$. According to the null-factor law, the roots obtained are $x_1 = 5$; $x_2 = 7$. It is often a problem with the negative numbers in this kind of factorization. In order to check if the solution is correct, multiplying the two factorized binomials by distributive law will show if the expanded polynomial is the same as the original quadratic equation.

Type 5. The fifth kind of quadratic equations is $ax^2 + px + q = 0$. Still, the methods of completing squares and quadratic formula can be applied for all type 5 equations, but factorization can also be used for solving this type of equations though it may be complicated and requires systematic work. Taking the example of $6x^2 + 5x + 4 = 0$, this quadratic equation can be solved by the factorization method as mentioned in study three in 2a according to Jackman (2005), but it is too complicated and troublesome. However, the use of a guess-and-check factorization method can sometimes be very efficient for solving the equation, making use of arithmetic skills and a good number sense.

Among these five kinds of quadratic equations, the first three types are easy to solve by factorization while types 4 and 5 are more difficult and tedious.

3) Applying the quadratic formula for solving quadratic equations

Solving quadratics equations by factorization is constrained within the simple quadratic equations over the whole integers and rational numbers as coefficients. What happens when quadratic equations have coefficients that belong to an irrational domain or big quantity?

The quadratic formula is a solution to such a situation. As mentioned previously, the French mathematician François Viéte discovered the standard quadratic formula which is applicable for all kinds of quadratic equations (Olteanu, 2007):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$$

This quadratic formula is believed to have its origin in ancient Babylon (Kvasz, 2006). The Greek mathematician Euclid (Kvasz, 2006) used a geometrical method to derive this quadratic formula (see Figure 2). In Swedish textbooks, this formula is written in PQ form by making $a = 1$ in the equation $x^2 + px + q = 0$, that is:

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \text{ with } p = \frac{b}{a} \text{ and } q = \frac{c}{a}.$$

Olteanu (2007) shows that Swedish students' difficulties in using algebraic symbols are no longer the problem. On the contrary, how to handle the parameters or coefficients in quadratic equations like $ax^2 + bx + c = 0$ and rewrite them in the equivalent form $x^2 + px + q = 0$ with p and q being real numbers becomes obstacles for the students when they study the algebra course at upper secondary level. In the USA, the standard quadratic formula is regarded as the standard method for solving quadratic equations (Obermeyer, 1982). Obermeyer presents a way of deriving this formula by completing squares in 11 steps. All the methods mentioned so far are based on square rules: $(a \pm b)^2 = a^2 \pm 2ab + b^2$. The standard method or PQ form might be regarded as an efficient and direct method, but it may lead students to solve quadratic equations in a mechanical way; besides there is also the problem with the troublesome symbol $\pm$ (Stover, 1978). Both Stover (1978) and Olteanu (2007) have suggested using a graph of quadratic functions to solve quadratic equations as an alternative method.

3.3 Factorization and polynomials in abstract algebra

The algebra content at Swedish upper secondary school includes interpreting, simplifying, reformulating quadratic expressions and solving as well as applying quadratic equations according to the mathematics syllabus (Skolverket, 2000). The algebra content in the investigated Swedish mathematics textbooks covers topics like polynomial, operational rules for computing polynomials, factorization of simple quadratic expressions by using square rules and the difference-of-square formula inversely, solving quadratic equations. How are quadratic expressions like polynomials and quadratic equations related to factorization? In order to find mathematics links among these topics and trace them back to abstract algebra, this part presents a mathematical review by referring to two books concerning algebra structure (Vretblad, 2000; Durbin, 1992) and a research study (Bossé and Nandakumar, 2005).

Factorization and polynomial equations

A quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$, can also be regarded as polynomial equation. Solving quadratic equations is related to finding a zero point[3] for the polynomial equation (Alfredsson, Brolin, Erixon, Heikne, & Ristamäki, 2007, p. 107).

The polynomial $ax^2 + bx + c$ is a second degree polynomial. A polynomial consists of coefficients in whole numbers, rational numbers, real numbers or complex numbers and variables (sometimes called unknowns) in different degrees as well as constants. In the general form of an n-degree polynomial, $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$, the numbers a_0. $a_1 \ldots . a_n$ are coefficients. If all a_k are zero for $k \geq 1$, then $f(x) = a_0$ is constant. If all a_k are zero, then we say the f is a zero polynomial. If f is a polynomial, the equation $f(x) = 0$ is called an algebraic equation or polynomial equation at n-degree (Vretblad, 2000). A solution or a root of this equation is a number α so that $f(\alpha) = 0$. Such a number is even called a zero point of the polynomial f which means when $x = \alpha$, the value of the polynomial is zero.

[3] *nollställe* in Swedish

Factorizing polynomials is analogue to factorizing integers but also related to the concept of ring in the field of abstract algebra structure. According to Durbin (1992), the definition of a ring is:

> A ring is a set R (any nonempty set) together with two operations on R, called addition ($a + b$) and multiplication (ab), such that each of the following axioms is satisfied: R with addition is an Abelian group $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$, there is an element $0 \in R$ such that $a + 0 = 0 + a = a$ for each $a \in R$, for each $a \in R$ there is an element $-a \in R$ such that $a + (-a) = (-a) + a = 0$, $a + b = b + a$ for all $a, b \in R$, multiplication is associative $a(bc) = (ab)c$ for all $a, b, c \in R$, and distributive laws $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$. (p. 110)

It can be noticed that representations of these operations are presented as operational rules called commutative, associative and distributive laws in the investigated mathematics textbooks, for example in *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007). There are different kinds of rings depending on what domain coefficients of polynomials belong to. The ring of integers Z consists of integers as elements, and factorization works for cases like $12 = 3 \cdot 4$. The different polynomial rings are, for instance, $Q[x]$: the ring of polynomials in one variable with rational numbers as coefficients, $R[x]$: the ring of polynomials in one variable with real numbers as coefficients, $C[x]$: the ring of polynomials in one variable with complex numbers as coefficients. In a ring you can multiply any two elements, but you cannot always divide. Factorization is a way to deal with this fact in order to exclude the elements from a ring. For example, in Z a division like 7/3 or $Q[x]$ a division like $(x^2 - 2x + 5)/(x - 3)$ cannot be performed since there is no element in the ring which can be multiplied by the denominator to construct the numerator. In such cases, factorizing cannot be done. The elements are excluded from the ring. In contrast, $(12/3) = 4$ and $(x^2 - 1)/(x - 1) = (x + 1)$ can be expressed by the factorizations $12 = 3 \cdot 4$ and $(x^2 - 1) = (x - 1)(x + 1)$.[4]

A ring consists of a set with two operations, which are a sum and a product of two elements as defined above. Any polynomial from zero to n-degree polynomials can be written or operated on through addition and multiplication of the elements over a field F including whole numbers or integers, rational numbers, real numbers and complex numbers. There is a unique monic polynomial (a polynomial with the leading coefficient 1, for example the monic polynomial $x^2 - 2x + 1$ which is the greatest common divisor of the product of any other two polynomials over a field F if not both the polynomials are zero polynomials. This unique monic polynomial can be irreducible (prime) or reducible, for example $x^2 - 2$ is irreducible over the field of rational numbers since there is no more divisor, but it is treated as reducible over real numbers since $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$. If a polynomial of degree at least one has no other divisors, then it is considered to be irreducible or prime (Durbin, 1992).

Any polynomial can be written as a product of other polynomials and the sum of a polynomial through polynomial division, that is: $f(x) = (x - \alpha)q(x) + r(x)$. The term r is a constant when its polynomial is zero degree. It can be noticed that $(x - \alpha)$ is a first degree polynomial. When the rest r is zero, the value of the polynomial $f(x)$ is zero, something which also is expressed as that the polynomial has a zero point when $x = \alpha$ (Vretblad, 2000). Taking the example $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$, both $(x - \sqrt{2})$ and $(x + \sqrt{2})$ are irreducible polynomials with degree one which implies that the polynomial $x^2 - 2$ has zero points when $x = \sqrt{2}$ or $x = -\sqrt{2}$. In this

[4] Fainsilber, personal communication, March 9, 2009

way, the polynomial equation $x^2 - 2 = 0$ can be solved by factorization and its roots are these zero points: $x = \pm\sqrt{2}$.

In this way, solving a quadratic equation can be handled through finding the first degree factors for the second degree polynomial equation and thereafter finding the roots of the quadratic equation. Therefore, the factorization method is one of the alternatives to solve quadratic equations. Using factorization for solving quadratic equations has rich connections with arithmetic operations, number concept and algebra structure.

For algebra beginners, factorization can be traced back to students' early years of mathematics education, when students not only work with multiplication but also on factoring integers, for example $56 = 8 \cdot 7 = 2 \cdot 2 \cdot 2 \cdot 7 = 4 \cdot 14 = 2 \cdot 28$. In this example, two operations are included: multiplication and division. Multiplication operation may help students to understand the multiplicative structure of the integers, while factoring integers may become meaningful when it is related to divisibility, divisors, and prime numbers. These two operational skills, or competences, are major techniques for simplifying fractions and finding common denominators. Being able to understand and operate factoring integers has possibly laid the essential grounds for algebra beginners to study factorization of polynomials.[5] Thus, it is possible to assume that factorization reunites students' early mathematics knowledge of factoring integers with later knowledge of algebra structure.

Factorability

In a study on factorability by Bossé and Nandakumar (2005), the probability of factorability is reasoned through investigating the range of integers as coefficients of quadratic equations. In a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, a, b and c are randomly generated integers within a determined range r such that $x \leq r \leq y$ where x and y are integers. When $b^2 - 4ac$ is a perfect square or where $\sqrt{(b^2 - 4ac)}$ is an integer, quadratics are factorable. The study shows that as the range for a, b and c increases, the probability of factorability of a quadratic with randomly selected integer values for a, b and c decreases. Bossé and Nandakumar have confirmed that as the range for coefficients expands to $-\infty < r < \infty$, the probability of factorability of a quadratic with randomly selected integer values for coefficients approaches zero. The data of the study collected from college algebra courses and textbooks demonstrates that about 15% of the quadratics with integer coefficients $-10 \leq r \leq 10$ are factorable. Thus, about 85% of all quadratics can not be factorized. In spite of the limitation of factorability, within the exercises concerning factoring practice, selected from 27 surveyed college algebra textbooks, about 94% of the problems were factorable. Within the textbooks, 55% of the quadratic expressions to be factorized are within the range [-10, 10]. Even if many quadratic equations are factorable, choosing the right factor pairs is time consuming, for example the equation $36x^2 + 59x + 24 = 0$ has to be factorized by choosing the right pairs of factors among nine pairs for 36 and eight pairs of factors for 24. Comparing the three different methods for solving quadratic equations in their study, Bossé and Nandakumar (2005) suggest that completing the square and quadratic formula are more effective, informative and useful than factorization.

[5] Fainsilber, personal communication, March 9, 2009

3.4 Conclusion

This chapter has presented core content in the investigated mathematics textbooks: completing the square, factorization, quadratic formula, and the relations between roots and coefficients as well as constant in a quadratic equation. Early ideas of algebra originated from geometrical ideas on finding the sides of a square or rectangle by Babylonian mathematicians. Using completing the square approach to solving quadratic equations has its roots in the geometrical method. Euclid developed the geometrical method and solved quadratic equations by making use of the equivalence relation between the completed square and a rectangle. In such a way, the quadratic formula derived through a geometrical method which was expressed in words by the Babylonians. It was al-Khwarizmi (Kvasz, 2006) who wrote the first text on solving quadratic equations, although he used the same geometrical method. In the late 16th century when algebra came into the symbolic stage, French mathematician Viète found the relationship between the solutions of the equation and the coefficients for first five degrees of equations with one variable (Derbyshire, 2006) and discovered the quadratic formula for solving quadratic equations written in algebra symbols (Olteanu, 2007). In the 17th century, algebra went into the abstract stage where functions and algebra structure became the main focuses (Derbyshire, 2006). Historically, algebra was developed from geometrical ideas to symbols and manipulation rules, and later to abstract axioms and structure. The process took about 4000 years (Kvasz, 2006). The core content presented in the textbooks has many similarities with the history of algebra.

The approaches of completing the square and quadratic formula are the common topics taught in school algebra (e.g. Allaire & Bradley, 2001; Bossé & Nandakumar, 2005). Using factorization to solve quadratic equations or simplifying quadratic expressions is taught in different ways in different cultures (Kemp, 2010). For example, there is a *guess-and-check* method based on observing the structure of quadratic equations or expressions and a *cross multiplication* method based on looking for factors of the coefficients (Kemp, 2010). Using *algebra tiles* as concrete material to teach students factorization is another method based on geometrical ideas (Leong et al., 2010). Factorization is not only used in elementary algebra but also applied in advanced algebra. In algebra structure, factorization is related to operations in a ring (Durbin, 1992). The principle to use factorization to solve quadratic equations is that a quadratic equation can be written into two factors by using the distributive law in reverse. This is called the null-factor rule (Alfredsson et al., 2007). Bossé and Nandakumar (2005) have categorized five different quadratic equations solved by the factorization method. The simple quadratic equations can be solved by using the square rule and the difference-of-two-squares rule inversely. But the general quadratic equations are difficult to handle by factorization. Then completing the square method and the quadratic formula can be used for all kinds of quadratic equations.

4. Research Reviews

This study is about analyzing algebra content presented in textbooks and involves three research areas: textbooks, algebra related to Pedagogical Content Knowledge; PCK (Shulman, 1986b) and algebra related to Subject Matter Content Knowledge; CK (Mishra & Koehler, 2006).

Algebra content knowledge related to this study has been presented in Chapter 3. In order to gain insight into research within these three areas, I have systematically searched articles and handbooks to review previous research. In this chapter, I will summarize my findings of previous research within the areas of textbook research and algebra teaching and learning.

This chapter consists of two parts including two research reviews. The first part focuses on previous studies on textbook research including mathematics textbooks. The second part focuses on previous studies on teaching and learning school algebra, in particular factorization and quadratic equations.

4.1 Review on previous textbook studies

This first research review begins with the relation between textbooks and teaching. The purpose is to show why I relate content analysis to teaching and subject content knowledge. Thereafter, a survey of previous research on textbooks will be presented by referring to Johnsen (1993) and Selander (2003) in order to get an overview of the field. Then the main part will focus on some previous textbook studies by relating to a teaching and learning perspective as well as subject content. The conclusion will be given after the first review on textbook research.

A short background of textbook inspection in Sweden

Sweden has a long tradition of examining and control of school textbooks by the state (Långström, 1997). Ever since the 17th century, school books have been examined and inspected by the Swedish government or by the church so that none of the books contained heresies. In 1868, all textbooks in geography and history for elementary schools were inspected. During the interwar period, the textbook issue was discussed and investigated.

In 1938, a government textbook commission was established. The inspection of textbooks was extremely extensive, detailed and covered most of the teaching aids including not only content and style but also price and even the quality of the paper. Working teachers and experts in the respective fields carried out the inspection and examined the books according to the directives stated by their commission. Textbook investigation by the government has continued. In 1974, the SIL (Statens institut för läromedelsinformation - “The Swedish Institute for Teaching Aid Information”) was established to examine the objectivity level of all teaching aids used in social studies. In the latter half of the 1980s, a few inspections on different themes were carried out. In 1991, the SIL ceased its work and the board of education took over. However, inspection of textbooks at government level in Sweden met its end in 1996. Today, the quality of textbooks is decided by the demands of the market (Långström, 1997).

4.1.1 How does a textbook relate to teaching?

Making clear what a textbook is and how textbooks relate to teaching and pedagogy is an important issue in my study. I choose to present the relationship in Section 4.1.1. To do this, I refer to a research paper (Stray, 1997), a previous study (Julin Svensson, 2000) and a survey (Selander, 2003) as well as some handbook studies (Doyle, 1992; Venezky, 1992).

In general, textbooks and other teaching resources are often included in the classroom contexts according to Dunkin and Biddle (1974, cited in Shulman, 1986a). Alongside other learning material, textbooks are used as instructional material in a particular subject (Gagné, 1977; Reints, 1997; Venezky, 1992). Belonging to the whole teaching process, teachers are provided with textbooks for planning and teaching lessons (Abell, 2007).

The relationship between textbook and teaching has been claimed to be curriculum related (Venezky, 1992). Venezky claims that a textbook is regarded as a replacement of the curriculum including subject content, hidden curriculum and pedagogical approaches since it offers arranged topics, didactic methods, and instructional manual for a particular subject. He argues that a textbook influences teaching, but also that curricula have strong influence on textbook content through the publishers. Textbooks connect knowledge domains to school subjects by transforming content into curriculum (Doyle, 1992).

Going to the details about a textbook, I find the connections among a textbook, a subject and pedagogical functions in its definition. The *Longman Dictionary of Contemporary English* (2005) defines that a textbook is “a book that contains information about a subject that people study, especially at school or college” (p. 1714).

Some researchers in the field of textbook research have discussed the issue of how to define “textbook.” To answer this question, Stray (1997) means that several answers can be taken into account. There are textbooks produced for use in instructional sequences, but on the other hand there are textbooks produced by authors who do not have such instructional sequence intentions. For example Shakespeare’s plays can be used in a classroom. Textbooks and schoolbooks can be distinguished: textbooks are designed and produced specifically for instructional use while schoolbooks also can be used for instructions but are less closely tied to pedagogic sequences. “Schoolbook is first attested in the 1750s, and more commonly from the 1770s. Textbook does not appear until the 1830s” (Stray, 1997, p. 57).

Textbooks have been related to teaching for a hundred years as a teaching instrument and they link teachers, students and the searched knowledge (Julin Svensson, 2000). Textbooks are regarded as part of teaching materials[6] (Julin Svensson, 2000; Selander, 2003). Teaching materials including books, instruments and other assistant materials are intended for students’ study in main school subjects in order to reach the set aim of the curriculum (Julin Svensson, 2000). In Sweden, textbooks have been categorized as basic teaching materials by SIL,[7] according to Selander (2003). The difference between textbooks and other teaching materials is that textbooks can be used both in and outside school while other teaching materials are restrained within school situation (Selander, 2003).

[6] *Läromedel* in Swedish
[7] *Statens institut för läromedelsinformation* - “The Swedish Institute for Teaching Aid Information”

The essential feature of a textbook is what is presented in the textbook, or in other words content in the textbook. Furthermore, content in the textbook is mainly expressed by texts. Such texts are regarded as pedagogical texts (Selander, 1997, 2003; Julin Svensson, 2000). Pedagogical texts differing from other kinds of texts are produced for institutionalized use in an educational system that has its own spatial, time, and social organization such as classes and lessons etc (Julin Svensson, 2000). By referring to Grepstad (1997), Selander (2003) emphasizes that pedagogical texts in textbooks or other teaching materials do not aim at creating new knowledge but reproducing available knowledge. He continues that they are structured with certain pedagogical needs to explain something; at the same time the knowledge represented by the texts should be able to be applied or controlled by teachers in a relatively easy way. Selander (2003) regards pedagogical texts as tertiary artifacts since they are produced for students, and even by students in a teaching situation. He points out that school knowledge depends mostly on textbooks that are treated as standard. In an example used by Selander (2003), he argues that textbooks' function is not just for mediating the facts of subject matter knowledge but encouraging active learning through reconstructing the knowledge with a pedagogical idea of utilizing the subject matter knowledge as a process rather than an amount of facts.

In this study, the analysis of mathematics textbooks focuses on algebra content presented by mathematical texts in the textbooks. In this meaning, the mathematical texts are pedagogical texts that have the pedagogical functions of explaining the knowledge of algebra.

4.1.2 Two influential surveys of previous textbook research

In this part I will present two surveys of textbook research made by Johnsen (1993) and Selander (2003) in order to get an overall picture of previous research in this field.

Johnsen (1993) uses the metaphor of a kaleidoscope to describe the complexity of a textbook since it consists of many different aspects related to curriculum, instruction, and knowledge. His survey covers research material from Germany, France, the Nordic countries, the UK and the USA, and includes history textbooks, grammar books, writing skill books and textbooks for social studies. By referring to Woodward, Elliott and Nagel (1988), Johnsen (1993) applies three main categories of textbook analysis approaches: process-oriented textbook research, use-oriented textbook research and product-oriented textbook research. The process-oriented approach consists of conceptualization, writing, editing, agreement (by the publisher), marketing, selection and distribution before it comes to a user such as a pupil. This type of research has been scarce. The use-oriented approach refers to studying the use of textbooks by teachers and students when textbooks are regarded as educational instruments in school. The research questions are about the textbooks' authority, accessibility and effectiveness. This kind of research is growing but is not as common as the content analysis. The product-oriented approach refers to the fact that textbooks are produced as a product intended to provide information about a particular subject. Research has been carried out in the areas of content analysis, the selection of material and readers' attitudes.

Johnsen finds that content analyses have dominated textbook research. He writes:

> Little research has been done on the writing, development and distribution of textbooks. Most of the literature consists of articles criticizing either the approval systems or the role of the publisher.
>
> Book use has received slightly more attention, but so far the primary focus has been on textbook analyses based on readability theories rather than on classroom surveys. The results are conclusive on several points: Textbooks have been and continue to be the most widely-used teaching aid. Although it is hard to pinpoint exactly how textbooks are used in the classroom, it is clear that practices vary considerably. The way in which pupils read and use textbooks has not yet been studied adequately, but existing reports tell of poor accessibility and questionable effectiveness. (Johnsen, 1993, p. 328)

Compared to conventional educational research, referring to Westbury (1990), Selander (2003) points out that textbook research has been invisible in spite of the textbook's important role in education. However, textbook research has started to get more attention during the last 15 years in many nations like Germany, France, Japan, England, Norway, Sweden, Austria, the USA and Australia (Selander, 2003). Selander categorizes textbook studies within three areas, very similar to the areas described by Johnsen (1993): process-oriented, for example textbook production, distribution and utilization; production-oriented relating to content, social and cultural aspects such as linguistics and didactical aspects; reception-oriented, for example, how students understand texts according to French researcher Alain Choppin (as cited in Selander, 2003). The survey shows that textbook production and distribution are the least studied area because of unwilling cooperation from publishers (Selander, 2003). Textbook research has been focused on studying textbook content, but studies on how they are used and how they can contribute to learning are needed (Selander, 2003). Within the product-oriented research area relating to textbook content, subject-didactical studies[8] and critical discourse analysis have been studied and developed, according to Selander. He explains that the subject-didactical studies treated as pedagogical research in Sweden, have involved subjects like history, geography, civics, physics, biology, religion, mathematics, literature and music.

One finding in the survey of research on textbook and teaching materials by Selander (2003) is that pedagogical texts today tend to be created with "learner-centered design (LCD)" (p. 217), which is considered a pedagogy against passive learning. In such pedagogical texts, tasks or assignments offered to students are not too complicated or difficult but designed at different levels in order to adapt various needs from students and encourage the student's learning (Selander, 2003). In contrast to the traditional and normative pedagogy, the pedagogy reflected in these kinds of teaching materials contains more activities and requires more cooperation (Selander, 2003).

The influence of textbooks and teaching materials on teaching has been studied within Swedish contexts (Englund, 1999; Johnsson Harrie, 2009; Julin Svensson, 2000). The role or function of textbooks in schools has been treated differently depending on subjects and teachers (Englund, 1999). Looking at a study by Gustafsson (1980), Englund (1999) points out that in the subjects of English and science courses, textbooks play an important role as information transformers providing plain facts and as organizers of the subject contents, while civics or social studies depend less on textbooks. The positive influence of the textbook may

[8] *ämnesdidaktiska studier* in Swedish

be that textbooks are helpful when applying teachers' teaching methods and content sequences while they may negatively affect students' learning from a textbook language and ideology aspect. Agreeing with some Swedish studies, Englund supports the idea that textbooks define teaching aims and what teachers present mostly comes from textbooks. Another function of textbooks is to keep students busy during the lesson. On the other hand, textbooks do not represent an independent agent. It is teachers who decide what they intend to use from textbooks and who find pedagogical aims and structures in textbooks. When teachers lack subject knowledge, they depend heavily on textbooks. Mathematics textbooks have a strong influence on both teachers and students because of its content and sequences. Similar to Englund's result, some American studies from the 1980s (Doyle, 1992) indicate that teachers' dependence on textbooks varies depending on the subject they teach.

To sum up, the general findings of previous research are that textbooks production and distribution as well as writing have been studied very little; most of the studies have focused on analyzing textbook content related to its subject matter knowledge, whereas textbook use in classrooms and how textbooks influence students' learning have not received much attention. (Johnsen, 1993; Selander, 2003). In the textbook content analysis research, the pedagogical idea related to learner-centered design built into the textbook has been found according to Selander (2003). My study can be positioned in the last category: textbook content analysis research but with focus on studying the embedded pedagogical content knowledge.

4.1.3 Review of previous research on mathematics textbooks

The review in this section is based on a study of previous textbook research. I carried out my search for literature about textbook research by using three databases and library resources. During the searching process, two areas were focused upon: textbook research in general and mathematics textbooks research in particular. In order to get a general insight into textbook research, the review covers both articles and dissertations related to different subjects including mathematics. Three categories (Johnsen, 1993; Selander, 2003), of textbook research were taken into account: textbook content, textbook use, and textbook production. In addition, I investigated if the studies involved analyses of pedagogical intentions in textbooks from a teaching perspective, and what analytic frameworks were applied for content analysis. Among the searched articles and dissertations, 21 studies were chosen for reading in detail and they include the different subjects: language learning (3); history (2); social studies (2); geography (1); science (3) and mathematics (8). Two of the twenty-one studies were excluded because of their irrelevance to content analysis of textbooks.

In the twenty-one studies, eight studies are related to mathematics textbook research but only one study (Jakobsson-Åhl, 2006) is about analyzing algebra content in textbooks. Analyzing mathematics tasks in the textbooks (Brändström, 2005) is a study within the area of textbook content analysis. Two of the eight studies are about both textbook content analysis and the use of the textbook in mathematics classrooms in relation to teaching (Johansson, 2006; Pepin et al., 2001). Comparison of mathematics tasks from the textbooks across different nations has been found among these studies (Vincent & Stacey, 2008). Of these twenty-one studies, 90% are textbook content analysis and a few of them are about both content analysis and the use of textbooks in relation to curriculum. Two textbook content analyses in mathematics and science (Chiappetta & Fillman, 2007; Vincent & Stacey, 2008) have applied TIMSS (The Third International Mathematics and Science Study) analytical criteria concerning conceptual framework for analyzing content in textbooks. Other research methods for analyzing textbook content involve variation theory (Jakobsson-Åhl, 2006), and a mix of quantitative and qualitative methods (Chiappetta & Fillman, 2007). However, there are no general findings among the twenty-one studies except that most of the studies are about textbook content analysis.

In the following part, the review of five relevant studies will be organized according to PCK-CK (Shulman, 1986b; Mishra & Koehler, 2008) principles and entitled *subject teaching and learning* (as PCK) and *algebra content* (as CK). I add the third title *content analysis framework* (related to PCK-CK) to the review since analytical frameworks are relevant for this study.

Subject teaching and learning

Johansson (2006) has studied mathematics textbooks in relation to curriculum and teaching. In her study, she examines Swedish textbooks and the national curriculum and then observes how the investigated textbooks are used in three Swedish classrooms. Johansson finds that the development of the curriculum is only partly reflected in the development of the textbooks. Teachers and students are the most crucial factors relating to mathematics textbooks. They are the users and readers, textbooks become artifacts in the mathematics classrooms. In her study, Johansson carries out the investigation of content analysis of a series of mathematics textbooks from Swedish lower secondary schools with focus on the influence of curriculum development in these textbooks. The content analysis consists of special mathematics topics from the mathematics textbooks for grade seven in lower secondary schools and the aim is to examine the link between the intended curriculum: “The analysis shows that there are very few instances in the textbooks where mathematics as a scientific discipline is discussed” (Johansson, 2006, p. 24).

When it comes to the use of mathematics textbooks, Johansson (2006) claims that mathematics teachers in Sweden have an influence on the development of textbooks because they decide which textbook they will use. Her finding reveals that teachers (two out of the three teachers in her study) use their textbooks as the main sources for both teaching and students’ individual work during the whole lesson in the classrooms (Johansson, 2006). The teachers’ presentations of mathematical definitions, rules and problem solving procedures as well as examples are portrayed in the textbook. Johansson also exposes that tasks and their constructions influence the teacher-student interaction. When the teacher and the textbooks have different answers, the teacher often avoids arguing against the textbook, which causes ambiguity and makes the students confused about how to understand the mathematics subject, according to Johansson.

Johansson’s study supports findings from another study (Gustafsson, 1980), that teachers depend heavily on mathematics textbooks in their teaching and that mathematics textbooks are the main sources of mathematical knowledge that teachers present. There is an authority status of mathematics textbooks even though mathematics textbooks have not emphasized mathematics as a scientific discipline (Johansson, 2006; Pepin, Haggarty & Keynes, 2001). Another factor is that mathematics textbooks reflect the development of curriculum and cover the same topics as in the curriculum (Johansson, 2006; Venezky, 1992; Pepin et al., 2001).

In an international textbook study (Pepin et al., 2001), content analysis of mathematics textbooks has been carried out in England, France and Germany associated with their use by teachers in mathematics classrooms at lower secondary level. The analysis in this comparative study considers four areas in terms of textbook content and structure: the mathematical intentions; pedagogical intentions; sociological contexts; and the culture traditions represented in textbooks. Their study aims at exploring the mathematics classroom teaching and learning culture against a background of a textbook analysis and teachers' use of them. The motivation behind their study came from findings of a TIMSS study that included large-scale cross-national analyses of mathematics curricula and textbooks in almost 50 nations. Schmidt et al (1997, cited in Pepin et al., 2001) pointed out that textbooks are shown to reflect official intentions of the national curriculum. They argued against research on text analyses that were distanced from their context of use. Thus, they chose to analyze textbooks not only in terms of their content and structure but also their use in classrooms by teachers and students.

Textbook analyses conducted by Pepin, Haggarty and Keynes (2001) focus on the mathematical intentions and pedagogical intentions implicated in the textbooks. The mathematical intentions cover three areas: what mathematics is represented in textbooks; beliefs about the nature of mathematics implicated in textbooks; and the presentation of mathematical knowledge. In their previous literature reviews, Pepin et al. (2001) put forward different views on mathematics taught in schools and mathematics in the academic world. Against the view of regarding school text material as a special version of mathematics, they seem to agree with Love and Pimm (1996) that both so called "real" or scholar mathematics as well as school mathematics with pedagogical intentions should be seen as different versions of mathematics for a particular purpose (Pepin et al., 2001, p. 4). However, this does not clearly show what their views of mathematics are when analyzing textbooks. Their beliefs in the nature of mathematics implicit in textbooks are based on Van Dormolen (1986), who means that acquisition of knowledge with activities in textbooks is one kind of goal and acquisition of process skills with content knowledge growth is another goal. With these two goals, mathematics textbooks may be written so as to explore and encourage students for acquisition of new knowledge; or in a way that puts focus on a large amount of exercises with few connections with concepts. Compared with the view of mathematics as rule-bound and convention-bound from Van Dormolen, Pepin et al. (2001) give another opinion on understanding according to Schmidt et al. (1996, 1997) associated with cross-nation research: understanding content has much to do with topic, developmental and cognitive complexities, especially the last complexity generalized through activities like recognizing, recalling, performing, solving and developing and so on. In such a way, Pepin et al's comparison study explores the diversity in textbooks in three countries.

The criteria applied for analyzing pedagogical intentions of textbooks by Pepin et al. (2001) are based on Van Dormolen's idea and mixed with those of Schmidt et al. In the previous studies review, Pepin et al. (2001) agree that textbooks are used extensively in the classroom and that textbooks characterize authorities in different ways such as legitimating knowledge, being major resources in the classroom, transmitting knowledge and so on. They claim that "teaching of the text has always been the teacher's primary function, with the teacher as mediator" (Pepin et al., 2001, p. 7). Accordingly, it is often teachers who decide which textbook to use, where and when to use it in the classroom. They argue that research on textbooks should put focus on the use of textbooks too, which may provide a representative picture of a country's education culture.

The research carried out by Pepin et al. (2001) is organized in three parts: analyzing mathematics textbooks in three countries; interviewing teachers about their views on using textbooks; and classroom observations. According to Pepin et al. (2001), the primary findings point out some differences and similarities between the three countries.

French textbooks consist of activities (small investigations and cognitive activities), essential exercises (essential parts for teaching and learning including working examples), and accommodating exercises (graduated in order of difficulty). The activities are intended to guide students to a new notion. French textbooks reflect the tradition of Piaget's constructivism and intend to develop students' mathematical thinking including exploring, understanding concepts and mathematical reasoning. All students had textbooks and the same textbook was used by all the students in the same year-group with the purpose that every student is offered the same opportunity, something which reflects the educational tradition of entitlement, an egalitarian view. Teachers relied on textbooks mainly in terms of exercises and cognitive activities, sometimes as an essential part.

German textbook investigations at lower secondary level consist of three different textbooks adapted for three different school forms. All the textbooks are constructed into two parts: introductory exercises and the main notion followed by an extensive range of exercises. The level of difficulty in complexity and coherence is high with respect to mathematical logic and structure, but representations show little variety according to Pepin et al. (2001). The textbooks were needed in every lesson, but teachers had to help low achieving students to understand the content in the books in some schools. Teachers used textbooks mainly in terms of exercises in school and for homework.

British textbooks seemed simple in terms of complexity and coherence according to Pepin et al. (2001). Straightforward questions are put forward before the worked examples. The taught method is applied in poor and "non-real" contexts in the form of "routine-type" (Pepin et al., 2001, p. 16). The contexts rarely required deeper levels of thinking. Students' access to textbooks was limited. Teachers used textbooks only for exercises and were less dependent on them than in the other researched countries, but they also at times tended to increase using textbooks because of lack of time. Teaching was student-centered and concerned a more individualistic approach.

Drawing upon the findings from Pepin et al. (2001), mathematics textbooks are designed and constructed differently depending on different pedagogical culture but they are generally applied for the pedagogical activity of doing exercises and mathematical activities. For example, French textbooks embed pedagogical ideas of encouraging students' thinking and reasoning mathematics. French textbooks emphasize that everyone should learn the same thing. German textbooks provide difficult and complicated mathematics tasks while British textbooks emphasize routine exercises without requiring deep level of thinking from students. Teachers in British schools depend less on mathematics textbooks. How the analysis criteria of mathematics content presented in the textbooks work in their study from methodology aspect has not been reported.

According to the findings of three mathematics textbook studies (Brändström, 2005; Pepin et al., 2001; Vincent & Stacey, 2008), there is an implied pedagogical idea that mathematics tasks or exercises offered by textbooks should not be too difficult or complicated for students.

A study carried out by Australian researchers applies the TIMSS Video Study criteria to analyze Australian eight-grade mathematics textbooks (Vincent & Stacey, 2008). The aim of the study is to compare mathematics textbooks' content (what is taught from the textbooks) with the findings of the TIMSS video study to determine whether the general picture revealed by the 1999 video study would arise if all lessons followed textbooks exactly. Vincent and Stacey (2008) point out that Australian mathematics classrooms heavily depend on textbooks and worksheets which take up 90 per cent of the lessons. Their purpose is also to find if it is possible to identify the differences between textbooks when they use the Video Study criteria. In their review of previous results from the TIMSS Video Study, they emphasize that the Japanese lessons have focused on making connections and so do their textbooks. Several negative trends in the development of mathematics learning have been noted in their review. For example, the US mathematics curricula of the 1970s called for teaching students to master basic mathematics procedures. Schoenfeld (2004, cited in Vincent & Stacey, 2008) argued that the focus on process without attention to skills deprives students of the tools needed for fluid and competent performance. The Australian researchers selected the 2006 best-selling eighth-grade textbooks in four Australian states. The investigated topics were addition and subtraction of fractions, solving linear equations and geometry concerning triangles and quadrilaterals. The researchers investigated the numbers of mathematics tasks in every book in their analyses. They classified the tasks according to five criteria: procedural complexity at three levels (low, moderate, or high procedural complexity), type of solving processes (using a procedure, stating a concept, making connection), degree of repetition, proportion of application (sometimes called real world) problems, proportion of problems requiring deductive reasoning.

Vincent and Stacey (2008) expose that lower procedural complexity tasks are in majority in most of the textbooks, but it does not necessarily imply lower quality of tasks in terms of challenging students to make connections or to reason or vice versa. There is a lower level of making connection tasks in several books. They argue that there needs to be a balance between acquiring mathematical skills and experiencing the processes enabling students to use mathematics. Advice is given by their study that using the percentage of problems in each category as the basic measure may be misguiding since time spending on different complex level of tasks varies much. Low-level tasks need less time than tasks with high procedural complexity or tasks that require students to make connections. Therefore, it is worth examining the percentage of time spent on tasks. The result also shows a broad similarity between textbook problems and the Australian Video Study lesson problems. They suggest that it is important that textbooks provide students with sufficient problems so that procedures may be practiced and become a secure part of a student’s mathematical toolbox. A certain level of repetition is useful. Their result does not show which tasks are so called good tasks and there is no evidence marking if a task provokes or does not provoke mathematical thought. They point out that a full range of task types and a balance task types are important for all the students, and that textbooks should have an accompanying teachers’ guide focusing on the pedagogical intentions of the textbooks material.

In contrast to the Australian study, Brändström (2005) exposes that the tasks in the textbooks used in Swedish lower secondary schools are not totally related to the educational demands and that the level of challenge is low. She has analyzed the levels of difficulty for tasks in mathematics textbooks from years 7 to 9 used for Swedish lower secondary school. The study focus is differentiation among presented tasks in the textbooks. In her analysis, Brändström uses two perspectives (Anderson and Krathwohl, 2001; Smith and Stein, 1998) to study the thought process expected of a student when solving a task.

Through describing the structure of one chapter in each of the analyzed textbooks, Brändström (2005) illustrates the strands of the different difficulty levels of the tasks in three textbooks and compares them with an applied analytical framework. She reveals that the three analyzed textbooks have a similar structure grouping tasks by difficulty levels, but the tasks have low challenge level that does not reach the educational demands. She suggests that the constructed analytical framework or tool can be used to study differences between tasks in mathematics.

Low level tasks offered in the mathematics textbooks are found in the three studies mentioned above (Brändström, 2005; Pepin et al., 2001; Vincent & Stacey, 2008). This exposes the pedagogical idea that everybody should learn something in their first step to study mathematics. Requiring students to perceive mathematical connections is found in the study by Vincent and Stacey (2008). Encouraging students' mathematical thinking is reflected in French mathematics textbooks, according to the study by Pepin et al. (2001).

Content analysis framework

This review has found that different kinds of analytical frameworks or criteria have been applied when analyzing mathematics content presented in textbooks. The analysis criteria applied for analyzing pedagogical intentions of textbooks by Pepin et al. (2001) are based on a view of mathematics presented by by Van Dormolen (1986) and blend with those of Schmidt et al. (1996, 1997) as mentioned before. They consist of three themes considering learners' understanding (Pepin, et al., 2001, p. 4):

1. Ways in which the learner is helped (or not) within the content of the text to learn the materials
2. Ways in which the learner is helped (or not) within the methods included in the text
3. Ways in which the learner is helped (or not) by the rhetorical voice of the text.

In short, the criteria concerns how content, methods and the rhetorical voice in textbooks are represented in regard to a learner. The criteria based on Schmidt et al. include investigating topics as well as developmental and cognitive complexities like recognizing, recalling, performing, solving and developing and so on (Pepin, et al., 2001). These criteria concern a learning perspective from a basic level to a complex level.

The analytical framework used in Brändström's study (2005) includes two perspectives. One perspective is based on the revised version of Bloom's taxonomy (1956) by Anderson and Krathwohl (2001). Bloom's taxonomy was originally "created to represent the intended outcome of the educational process and categories the students' behavior" (ibid., p. 27). Bloom's taxonomy has been regarded as a framework for classifying what teachers expect students to learn as a result of instruction according to Krathwohl (2001) (as cited in Brändström, 2005). According to her interpretation of Bloom's taxonomy, Brändström (2005) mentions three domains of educational activities identified in the taxonomy: cognitive, affective and psychomotor with focus on the cognitive domain.

The cognitive domain refers to practicing activities demonstrated by knowledge recall and intellectual skills (e.g. understanding ideas and applying knowledge). There are six hierarchical categories in this domain beginning from the simple behavior and building to the most complex: knowledge, comprehension, application, analysis, synthesis, and evaluation. Brändström (2005) claims that a student performing at a higher level demonstrates a more complex level of thinking. However, criticism of using the taxonomy arises during its application such as difficulty in interpreting the categories; the independence of content from process; and categories isolated from any context (Brändström, 2005). Considering the revised version of Bloom's taxonomy, Brändström (2005) emphasizes the importance of the categories consisting of remembering, understanding, applying, analyzing, evaluating, and creating. She claims that the last two categories are interchanged with the orders because of increased complexity. These categories indicate students' thinking at different levels from the low level remembering to the highest level creating. She compares these categories with another perspective from Smith and Stein's framework (1998) to study the thought process of a student when solving a task, pointing out four similarities between them: i) memorization ii) procedures with connections to concepts or meaning iii) procedures without connections to concepts or meaning iv) doing mathematics.

Brändström (2005) adapts Anderson and Krathwohl's categories and Smith and Stein's framework in creating her analytical framework. She investigates the processes and demands of the tasks in the textbook according to four perspectives: pictures (none; decorative; functional); operations (one operation or more than one operation); processes (remembering, understanding, applying, analyzing, evaluating, and creating); and demands (memorization, connections, no connections and doing mathematics).

In agreement with Pepin et al. (2001), Brändström (2005) considers the development of learning a process from a low to a high level. Analysis criteria related to process and demands of mathematics tasks have also been applied by Vincent and Stacey (2008). However, they have further added the tasks' procedural complexity, real world application, and demand of deductive reasoning to their criteria. Thus, I have found that content analysis of mathematics tasks or exercises in textbooks in these three previous studies focuses on the cognitive domain (Brändström, 2005) from a learning perspective in terms of pedagogy. Analyzing mathematics tasks also takes mathematics application into account.

When textbook analysis concerns content presented in the textbook as a subject matter, the analytic framework applied is of a different kind. Chiapetta and Fillman (2007) focus their content analysis of textbooks on subject matter related to concepts. They study five high school biology textbooks used in the United States. They claim that the role of textbooks in the US educational system is very important since textbooks help define school subjects and represent school disciplines to students. They find that science textbooks are often used as the primary organizer of the subject matter and 90% of secondary school teachers use science textbooks for classroom instruction and homework. Their study relates to four themes of the nature of science: science as a body of knowledge; science as a way of investigating; science as a way of thinking; science and interactions with technology and society. The aim of their study is to see if biology textbook authors and publishers responded to the recommendations of national reform committees and scholars with regard to teaching students a more authentic view of the nature of science. Chiapetta and Fillman (2007) claim that using a conceptual framework to guide textbook analyses is the most critical aspect. They take science content and recommended science instructional criteria into account when considering both the content and the instructional approach from a teaching perspective.

Other analyzing criteria used by Chiapetta and Fillman (2007) are based on a study carried out by Valverde et al. (2002) who have analyzed 630 mathematics and science textbooks throughout the world as part of the Third International Mathematics and Science Study curriculum analysis. Chiappetta and Fillman (2007) consider the following criteria:

a) physical features (number of pages and graphics)
b) textbook structure (sequencing content)
c) content presentation (coherence, fragmentation, and complexity)
d) performance expectations (reading, recall of information, answering questions, and engaging in hands-on activities)
e) lessons (the text segment devoted to a single main topic)
(Chiappetta & Fillman, 2007, p. 1853)

Chiappetta and Fillman (2007) also find that the American textbooks contain more pages and topics than those in other countries and have a large percentage of fragmented themes. The content analysis method applied by Chiappetta and Fillman (2007) is to code units according to four themes: knowledge, investigation, thinking, and science/technology/society. They argue that using the conceptual framework to analyze biology textbooks has offered more specific and detailed analyses when examining, for example, the authenticity of science in a textbook, the tentative nature of science, and so on. They find that the present biology textbooks contain enough information necessary to comprehend the fundamental ideas of the topics under study. They point out: “the textbook itself is a direct and concrete reflection of how that publishers and author choose to represent the nature of science” (Chiappetta & Fillman, 2007, p. 1864).

Differing from the previous mathematics textbook research mentioned before, Chiapetta and Fillman (2007) analyze the subject content presented in the textbooks, which include not only tasks but also content presentations. By focusing on biological concepts in the science discipline, they investigate the content structure and presentations in relation to topics and sequences. Thus, they analyze the textbooks from a teaching perspective and subject matter knowledge (Shulman, 1986b).

Algebra content

Algebra content analysis was found in one textbook study (Jakobsson-Åhl, 2006). Jakobsson-Åhl (2006) examines algebra content presented in two sets of upper secondary mathematics textbooks published in Sweden during the period 1960-2000. She has investigated literal calculi and algebraic theory as well as explicit descriptions of the concept of algebra in general in eleven algebra textbooks. Her investigation of the books includes algebraic definitions, descriptions, worked examples and exercises according to three categories: the Pre-New Math, New Math and Post-New Math eras. The methodologies adapted in her study are phenomenography and hermeneutics with the intention of describing the variation of school algebra in the textbooks from a second-order perspective. The aim of the study by Jakobsson-Åhl (2006) is to gain an insight into the variation of algebra in the period 1960-2000 in mathematics textbooks and their revisions when affected by curricular reforms. Her focus is on analyzing algebra content in school mathematics from four different points of view on algebra: operational symbolism; algebraic way of thinking; generalized arithmetic; algebra structure.

Jakobsson-Åhl (2006) has found a great change of algebra presented in the textbooks over the years. She points out that the pre-new math textbooks emphasize algebraic manipulation by use of literal symbols and algebra expressions, and that operational symbols are the central feature in these textbooks. Such complicated algebraic manipulation disappeared in the 1968 textbooks. Tabular and graphical representations have since then dominated and number structures became important. She continues that there was a shift of focus from algebra expressions to algebraic structures in the new math era. The algebraic structure approach disappeared in the 1970s. Instead, the use of numerical examples increased in the post-new math era while other changes in this period were that the real-life word problems appeared and equations were linked to the idea of relations and patterns. She also points out that there is a shift from algebra manipulations to multiple representations like functions, and tasks are related to real world and social contexts. Jakobsson-Åhl (2006) claims that the treatment of functions does not belong to the realm of algebra in Swedish textbooks and some of the specific aspects that traditionally belonged to algebra are now treated in the theory of functions. She argues how removing literal symbols takes away the opportunity to do algebraic manipulation. She suggests that algebra at the upper secondary level would benefit from being more closely related to requirements from universities. The importance of basic mathematics procedural skills are emphasized by Jakobsson-Åhl (2006) in her study.

Accordingly, algebra content in Swedish textbooks has been changed by moving focuses over time. Old textbooks put focus on algebra manipulation by use of algebraic symbols and rules. Algebra structures were also emphasized in older textbooks. When the complicated algebra manipulation and algebra structures disappeared from the old textbooks, table and graphical representations were greatly applied in the new textbooks. Numerical examples were employed and multiple representations like functions appeared in the new textbooks. Relating algebra to numbers, patterns, graphs and real-life situations became the focus of the new textbooks.

4.1.4 Conclusion on previous textbook research

Textbook research has mostly focused on content analysis while the use of textbooks related to teaching and learning has not received much attention in the field (Johnsen, 1993; Selander, 2003). The previous textbook research in this review belongs to a content analysis category (Johnsen, 1993; Selander, 2003) with two studies (Johansson, 2006; Pepin et al., 2001) involving the use of textbooks. Findings support earlier results that textbooks are teaching and curriculum related, for example, topics presented in textbooks often cover what teachers present in classrooms (e.g. Julin Svensson, 2000; Doyle, 1992). In the subjects of mathematics and science, textbooks are used extensively in teaching for organizing mathematical activities and providing students with mathematics exercises and teachers with teaching instructions (Johansson, 2006; Pepin et al., 2001; Chiappetta & Fillman, 2007). Textbooks embody pedagogy regarding both teaching and learning perspectives, for example subject content is organized in sequences (Chiappetta & Fillman, 2007); mathematics exercises or tasks are designed at low level in order to encourage everybody to learn something (e.g. Brändström, 2005). It has been criticized that some textbooks provide students with low-level tasks and lack motivating students' mathematical thinking (Brändström, 2005; Pepin et al., 2001). On the other hand, it is argued that providing students with low-level problems containing conceptual connections is necessary since students need opportunities to practice essential mathematical procedural skills (Vincent & Stacey, 2008). Mathematics textbooks reflect different pedagogical traditions depending on in which culture they are produced (Pepin, et al., 2001), for example French textbooks aim at fostering students' mathematical thinking and reasoning due to the influence of Piaget's constructivism.

In this review on textbook research, I have found that analytical frameworks or criteria applied in the four studies (Brändström, 2005; Chiappetta & Fillman, 2007; Pepin et al., 2001; Vincent & Stacey, 2008) partly include a cognitive perspective regarding thinking processes:

remembering, understanding, applying, analyzing, connecting, evaluating and creating based on Bloom's taxonomy (1956). These criteria are applied when analyzing tasks or exercises in textbooks. Criteria for analyzing subject content presented in textbooks utilize a conceptual framework and focuses on aspects such as content sequences, coherence and topics (Chiappetta & Fillman, 2007). In such a way, these criteria relate to a teaching perspective in textbook content analysis. There seems to be a relationship between analytical criteria or framework and content parts (presentations and exercises) in the textbook. However, application of PCK-CK (Mishra & Koehler, 2006; Shulman, 1986b) as analytical framework or criteria is absent from the previous studies on mathematics textbook analysis according to this research review.

4.2 Research review in the field of school algebra

In this part of the chapter, I will present a literature review of previous studies in the field of teaching and learning algebra. I first summarize a review written by Carolyn Kieran (2007) who gives an overall view in the field of school algebra. Then I will present a review of previous studies related to teaching and learning factorization and quadratic equations. A conclusion will be drawn at the end of this part. The aim of this part of the literature review is to find what previous studies have shown in the field of algebra teaching and learning in relation to pedagogical content knowledge (Shulman, 1986b) and content knowledge (Mishra & Koehler, 2006). The review involves both teaching and learning algebra since both of them are related to algebra content knowledge of factoring quadratic expressions and solving quadratic equations.

4.2.1 A general overview of learning and teaching school algebra

Kieran (2007) claims that research topics on school algebra in secondary schools have been discussed extensively in education for two centuries. For example, from the 1800s to the 1900s, procedures for manipulating symbols had great impact on school algebra. From the 1970s, focus has moved to algebra meaning for students. Research in algebra learning and teaching has gone on for decades in different fields such as: skill-based approaches during the 1960s and 1970s, influenced by Piaget's ideas on cognitive development; algebraic letters and structures during the 1970s and 1980s; and socio-democratic and computing technology influences since the 1980s. With the increase in digital tools in algebra learning, classroom discourse studies increased. At the same time, the mediating role of cultural tools drew research attention.

Kieran continues that research in the field of school algebra has had its emphasis on students' learning and understanding of algebraic concepts, ideas and methods for the beginners of algebra, or on the transition from arithmetic to algebra, but less focus on algebra teaching. Research studies conducted on algebra learning among the upper secondary and college level students are fewer compared to elementary and lower secondary level students. (Häggström, 2008; Kieran, 2007). Kieran reveals that algebra content has been taught with different focuses ranging between traditional and reforming ways. For example, the Japanese curriculum emphasizes symbolizing mathematical relationships from elementary level but with higher requirements at secondary level. Some other countries introduce algebra within the context of problem situations, sometimes including traditional word problems and sometimes realistic modeling problems which less emphasize symbolic manipulation.

Students from some countries like China, Russia, Singapore and South Korea, study algebra involving not only the development of algebraic reasoning and generalization, but also the use of algebraic symbols and solving of equations as early as in the fourth grade (Kieran, 2007).

Considering algebra content knowledge, Kieran (2007) categorizes research studies of school algebra according to three areas regarding algebra as an activity: *generational*, *transformational*, and *global/meta-level*. *The generational activities* of algebra refer to the forming of the expressions and equations that are the objects of algebra. For example, equations representing problem situations containing an unknown; expressions derived from geometric patterns or numerical sequences; expressions of rules governing numerical relationships. *The transformational activitie*s mainly refer to rule-based activities including, for instance, simplifying expressions, factoring, expanding, substituting, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations, working with inequalities or equivalent expressions, and so on. The transformational activity is not just training of skills but includes conceptual elements that arise during the process of learning transformations. *The global/meta-level activities* refer to problem solving, modeling, working with patterns which can be generalized, justified and proved, making predictions and conjectures, looking for relationships or structure and so on. Sometimes, the last kind of activities can be done without using any letter-symbolic algebra at all.

According to Kieran (2007), at the upper secondary level, research studies in the area of *general activities* focus on the letter-symbolic form including structure, parameters, multiple representations like function-graphical representations, and so on. In the area of *transformational activity*, research studies include equations and inequalities. Kieran points out that linear equations have received a great deal of research attention, but that quadratic equations have not–with the exception of a few studies. However, factoring expressions (factorization) have received research attention as a result of the emergence of computer algebra system (CAS) technology. This kind of factorization refers to factoring high degree polynomials which are higher than the third degree. There are also studies on integrating graphical and symbolic representations when, for example, finding roots of quadratic functions, solving linear and quadratic equations or translating quadratic functions. In the area of *global/meta-level activities*, studies involve problem-solving related to technological tools and modeling by using computer software systems. The role of realistic word problems in algebra instructions are problematic since research studies reported disparate results; some researchers reported advantages of experience with realistic word problems and modeling situations in algebraic activities while others did not (Kieran, 2007).

In research studies on algebra teaching, Kieran (2007) found that research studies on teaching algebra were less emphasized compared to research on learning algebra until the early 1990s, but there has been an increase in this field since then. She points out that some studies focused on what teachers of mathematics knew rather than what the teachers did in the classrooms. She criticizes that research studies have involved exploring algebra teachers' knowledge and beliefs such as teachers' pre-existing views and expectations in teaching practice but focused little on the learning process of algebra teachers. Kieran also puts forward the contrasting views revealed by different studies. One of them, conducted by Even (1990, as cited in Kieran, 2007), exposed the teachers' lack of understanding the concept of function and suggested that pedagogical content knowledge should include different representations, alternative ways of approaching the concept, a basic repertoire of examples, knowledge and understanding of the concept, and knowledge of mathematics; while another study conducted by Nathan and Petrosino (2003, as cited in Kieran, 2007) found that strong content knowledge among algebra teachers caused their teaching to heavily depend on the structure of mathematical domain rather than the actual ways in which students think. In spite of the imbalance in views from different studies, the overall view was that both knowledge and beliefs shaped teachers' teaching practice. What Kieran wants to emphasize is that some teachers have good pedagogical knowledge but lack of subject matter knowledge while some other teachers have good subject matter knowledge but lack of pedagogical knowledge of student learning.

Kieran (2007) points out that the gap between research on the learning of school algebra and the teaching of school algebra still remains. "However, researchers still know relatively little about what constitutes effective algebra teaching and how algebra teachers learn to develop their craft" (p. 749). Teachers' knowledge of students' actual algebraic thinking is limited according to research studies.

To sum up Kieran's review, previous research studies on teaching and learning school algebra have covered many areas in algebra as disciplinary knowledge, both concerning beginners and advanced learners. Previous research on algebra teaching seems to focus mostly on teachers' knowledge and beliefs but less on algebra teaching. It has also been found that few studies are carried out concerning quadratic equations. Research in the area of factorization using a paper-pencil technique is not found in Kieran's review. Kieran (2007) has mentioned that manipulating algebraic rules and symbols from early school years reflect cultural differences from different countries, something which may answer my question in the beginning of the introduction in this thesis: why factorization is not a focus when teaching how to solve quadratic equations in Swedish upper secondary classrooms. Studies on applying pedagogical content knowledge as a theoretical frame to analyze mathematics material including teaching instructions and mathematics textbooks are not included in her review.

PCK in algebra teaching

Ferrini-Mundy et al. (2003) have carried out a research project of studying teacher knowledge relevant for teaching school algebra in order to conceptualize knowledge for algebra teaching. They developed a two-dimensional analytic framework with three additional overarching categories for analyzing teachers' PCK in algebra teaching. The aim of the research is to find connections between teachers' mathematical knowledge and student outcomes in algebra through using and experimenting on the designed framework. Their empirical work includes interviews of algebra teachers, analyses of algebra instructional materials and analyses of videotaped algebra lessons. Based on teachers' and researchers' experiences as well as previous literature, the framework is designed and organized as a two-dimensional matrix with a third aspect containing three overarching categories to permeate the matrix.

In their framework, Ferrini-Mundy et al. (2003) combine one dimension containing categories of knowledge of algebra for teaching with another dimension containing tasks of teaching that identify actions in which teachers use mathematical knowledge. The three overarching categories labeled decompressing, trimming and bridging have an effect through all elements of knowledge of algebra for teaching.

In the first dimension containing categories of knowledge, Ferrini-Mundy et al. (2003) apply six categories. The first out of the six categories is teachers' algebra *core content knowledge* including algebra concepts such as variable, equation, expression, slope, and linear function; procedures such as solving linear equations, factoring, or simplifying expressions; and algebra connections or relationship. The second category is *representation*, which refers to various forms and models for concepts and procedures such as function graphs, algebra tiles, tabular or verbal descriptions of relationships between variables. The third category is *content trajectories*, which refers to organizing content in a particular order by knowing the origins and extensions of core concepts and procedures in order to support student learning. Using alternative teaching approaches and choosing powerful examples are included in content trajectories. The fourth category, *applications and contexts*, refers to whether teachers are aware of realistic mathematics education (Freudenthal, 1991) and provide students with real world problems related to different contexts and situations. The fifth category, *language and conventions*, refers to syntactic knowledge of the discipline of mathematics related to mathematics conventions, axiom, alternative terms or definitions of algebra concepts and so on. For example, knowing the operational order for multiplication of two binomial expressions as FOIL (first, outside, inside, last). Finally, the sixth category, *mathematical reasoning and proof*, refers to knowledge of the special vocabulary of reasoning, the ability to use various proof techniques within an axiomatic system to make convincing arguments, and the knowledge of standard conventions for algebraic arguments or justifications. For example, teachers should know that the processes used in simplifying algebraic expressions rely on the properties of rings of polynomials, including distributivity of multiplication over addition, additive inverses and associativity.

The dimension of the teaching tasks is categorized from teaching practices according to Ferrini-Mundy et al. (2003). The framework containing the two dimensions of their study is used as a guide for analyzing instructional materials, interpreting teacher interviews and for assessing teachers' knowledge in the project. One of the findings of their research points out that keeping a subject matter as focus for teaching is challenging. They have found that the framework lacks distinctions between mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, curricular knowledge, pedagogical content knowledge and so on, but also that the most useful models reflect an intertwining and integration of those areas. It is impossible to take apart the categories of knowledge from tasks used for teaching practices in the empirical work. However, the framework used for the research shows potential for measuring teachers' knowledge of algebra for teaching, although it needs to be further developed (Ferrini-Mundy et al., 2003).

4.2.2 Previous studies on teaching and learning factorization and quadratic equations

I have carried out a literature review study of research on algebra in general and on factorization of quadratic equations in particular, with focus on studies related to teaching and learning algebra. This work has been done through the use of three scientific databases and a search engine. In total, 113 articles have been investigated. Since the focal content of my empirical study is different approaches to solving quadratic equations, I have limited the investigation to this area after an extensive overview.

From the extensive searching of previous studies, I have found that the following aspects have been studied: algebraic thinking and reasoning (e.g. Johanning, 2004); understanding the transition from arithmetic to algebraic reasoning (e.g. Linchevski & Herscovics, 1996; Van Dooren, Verschaffel, & Onghena, 2002; Warren, 2003); conceptual understanding, such as symbols and variables (Asquith, Stephens, Knuth, & Alibali, 2007; Berry, Fentem, Partanen, & Tiihala, 2004; Hough, O'Rode, Terman, & Weissglass, 2007; Olteanu, 2007); and structural reasoning, such as equivalence and equal signs (Asquith et al., 2007; Hamdan, 2006). Areas reflected in those studies include symbols, equivalences, linear equations, graphs, use of calculators, algebraic concepts, algebraic thinking, and transition from arithmetic to algebra. These areas have been mentioned in Kieran's review, discussed in section 4.2.1.

Regarding algebra content knowledge related to quadratic expressions and equations, I have found that topics such as simplifying quadratic expressions and solving quadratic equations are often discussed as teaching ideas and strategies, shared by mathematics educators in different countries. Most of the articles related to quadratic equations found in the search are reports on teachers' teaching ideas or experiences rather than research findings. The topics from a mathematics teaching perspective include, for example, using geometric approaches to solve quadratic equations (Allaire & Bradley, 2001); utilizing completing squares (Vinogradova, 2007); factoring quadratics (Leong et al., 2010; Kotsopoulos, 2007; Rauff, 1994) and using factorization, completing the square, and graphical methods to solve quadratic equations (Bossé & Nandakumar, 2005; MacDonald, 1986; Vaiyavutjamai & Clements, 2006). It is surprising that only a few of these topics relate to research studies (e.g. Bossé and Nandakumar 2005; Leong et al., 2010; Vaiyavutjamai and Clements 2006).

Factorization has been discussed as a difficult aspect of algebra learning among students because of the limitation of memory (Kostsopoulos, 2007). Kostsopoulos (2007) links cognitive reasoning to pedagogical problems appearing in the mathematics classrooms in her article. Based on her teaching experiences, Kostsopoulos is aware of the fact that her pedagogical strategies lack insight into students' difficulties of multiplication and factorization of quadratics. She assumes that "students' problems with factorization and with identifying varied representations of the same quadratic relationship" may be linked to the ways in which the brain constructs cognitive representations (p. 19). Factoring of quadratics is the rewriting of higher degree polynomials as a product of lower degree polynomials. It requires students to have both a conceptual understanding of multiplication of polynomials and the effective procedural knowledge of basic multiplication facts. Phenix and Campbell (2007) declared that the order matters in the brain's ability to relate to number facts. From Phenix and Campbell's point of departure, Kostsopoulos (2007) conjectures that students' ability to access the appropriate long-term semantic memory is limited when they are confused by mixing the order and varied forms of quadratics. She provides a possible explanation as to why factorization is difficult to learn.

Teaching factorization based on students' own construction of the concept is a pedagogical approach used in one study (Rauff, 1994). With Von Glasersfeld's radical constructivism and Peter Gärdenfors' epistemic semantics theory as his theoretical frameworks, Rauff (1994) claims that belief-based teaching can be successful in teaching factorization. From the point of view of constructivist learning, Rauff (1994) emphasizes that learning occurs when students construct their own beliefs and knowledge of mathematics and later change their belief set. Referring to belief set theory from Gärdenfors (1988), Rauff (1994) points out that "a belief set is expanded when a new belief is added to it, and contracted when a proposition is no longer believed and is removed from the set" (ibid, p. 421). He continues that expansion, contraction and revision are three components to comprise basic mechanisms of changes of belief. When examining and comparing students' (ages 18-20) own definitions of factoring, Rauff (1994) analyzes the errors of their definitions with constructivism and epistemic semantics as analyzing tools. The result of his study exposes that the approach to teaching factoring, which relies heavily upon what a student believes about factorization, is quite productive in two ways. First, it reveals the source of nonstandard factorization; second, it provides a starting point for modification of the student's underlying conceptions of factorization. The pedagogy of belief-based teaching is suggested by Rauff (1994) to teaching other aspects of algebra such as polynomial multiplication.

Teaching to use a factorization approach to solve quadratic equations is regarded as an improper pedagogy by Bossé and Nandakumar (2005). Bossé and Nandakumar argue, on the basis of investigations of college courses and 27 college algebra textbooks, that the employment of factorization is not efficient compared to utilizing the quadratic formula or completing the square. They have found that only 15% of quadratics with integer coefficients at range of [-10, 10] among problems and examples from the textbooks are factorable. The probability of factorability of a quadratic with randomly selected integer coefficients from the textbooks was small. Bossé and Nandakumar (2005) criticize the teaching of factorization prior to other methods according to the conventional curriculum based on NCTM recommendations in US schools. They argue that it “entertains a false dichotomy, pitting mathematics against pedagogy” (ibid, p. 147). By demonstrating the strengths of using the techniques of completing the square and the quadratic formula as well as solving quadratic functions in graphs, Bossé and Nandakumar declare that these methods are useful for all kinds of quadratic equations and more efficient and informative than factorization which “is only appropriate for quadratic equations with rational roots” (ibid, p. 151).

Not only is factorization viewed as less useful by the study mentioned above, another study (Vaiyavutjamai & Clements, 2006) claims that using a factorization approach to solving quadratic equations does not help students to understand quadratic equations. Vaiyavutjamai and Clements carried out a study involving 231 students from two government secondary schools in Thailand. The aim of their study was to investigate how traditional lessons on quadratic equations–in particular when teaching grade 9 students to solve quadratic equations by factorization, by completing the square and by the quadratic formula–influence students’ understanding of quadratic equations. Their study consisted of 18 lessons of 50 minutes each with observations recorded on audiotape, pre- and post-teaching tests as well as 18 interviews for exploring students’ understanding of quadratic equations and their unknowns. They point to the lack of research on the learning of quadratic equations in association with understanding variables in quadratic equations. Vaiyavutjamai and Clements (2006) declare that “[s]tudent thinking in such contexts appeared to be dominated by a need to achieve procedural mastery, and usually there was no guarantee that relational understanding was achieved” (p. 49). With Skemp’s (1976) instrumental understanding and relational understanding as their theoretical framework, they aim to investigate if a traditional teaching approach does improve students’ relational understanding of quadratic equations.

Vaiyavutjamai and Clements (2006) reveal misconceptions regarding variables as obstacles for the students in understanding quadratic equations. The students have difficulties in discerning $x^2 - 8x + 15 = 0$ and $(x - 3)(x - 5) = 0$. These two equations are actually equivalent. The factoring form is just another form of the quadratic equation. The students do not think that x in $(x - 3)(x - 5) = 0$ represents different variables. They do not really understand the null-factor law (using factorization approach to solving quadratic equations). Many students obtained correct solutions but had serious misconceptions about what quadratic equations actually are from a mathematical point of view. Their study also exposes that the traditional teaching approach may improve students’ rote knowledge and performance skills but do not help their relational understanding of quadratic equations. They suggest teaching quadratic equations within the teaching of functions using modern technology such as graphic calculators.

In contrast with the two previously described studies, Leong et al. (2010) argue that factorization can be taught using a concretizing approach through applying algebra tiles or geometrical representations. In order to make a change of teaching factorization in the traditional way, they have carried out a lesson study including pre- and post-tests. Their study result shows the improvement of students' learning of factorization through the application of algebra tiles in teaching.

Research on the use of graphical representations to solve quadratic equations is found in a Swedish study (Olteanu, 2007). Olteanu (2007) focuses on algebra content analysis of learning quadratic equations and functions in algebra classrooms. She uses variation theory (Marton & Booth, 1997) as an analytical framework. The aim of her study is to analyze and explain the handled and learned object related to conceptual understanding of the algebraic symbol x in quadratic equations and functions. She finds that it is essential that students develop their ability of discerning three concepts–parameters, the unknown quantity and function–in order to understand the relation between quadratic formula, a quadratic equation and a quadratic function. Using graphical representations helps students to understand quadratic equations. She suggests that the extreme point of a quadratic function could be handled by using the derivate, which may be more comprehensible for students.

4.2.3 Conclusion of previous research on teaching and learning algebra

Kieran's (2007) summary of research on algebra teaching and learning shows that school algebra has been studied considering aspects of algebra generation, transformation, and modeling. At upper secondary level, research focus has been on algebra structure, parameters, multiple representations and the use of visualizing graphical representations. Research studies on linear equations have received great attention, but no studies have been done on quadratic equations and factorization with paper-pencil techniques. Factoring of high degree polynomials has received attention because of the use of computer algebra system. There is no common agreement about the role of real world problems in algebra teaching. Algebra learning has been much more widely studied than algebra teaching. Research on algebra teaching has been focusing mostly on teachers' knowledge and beliefs but less on algebra teaching. The same algebra content has been taught with different focuses depending on which culture or country it is taught in.

My review of previous research in this field has found different pedagogical views on teaching students to learn factorization. Factorization has been considered as procedural focused rote knowledge and less efficient (Bossé & Nandakumar, 2005; Vaiyavutjamai & Clements 2006), but it has also been shown that factorization can be taught by concrete geometrical representations to bridge the learning of abstract algebra symbols (Leong et al., 2001). Belief-based teaching as PCK for teaching factorization is found productive (Rauff, 1994). PCK has to be studied together with subject matter content knowledge both in general and in a teaching context (Ferrini-Mundy et al., 2003). Solving and understanding quadratic equations with a graphical approach has been suggested but needs further research (Bossé & Nandakumar, 2005, Olteanu, 2007). Previous research on teaching and learning quadratic equations is scarce. Vaiyavutjamai and Clements (2006) find that being able to solve quadratic equations procedurally does not improve students' understanding of quadratic equations. Application of completing the square and the quadratic formula methods for solving quadratic equations is recommended since they are more efficient than factorization, according to Bossé & Nandakumar (2005).

5. Analyzing procedure and methods

In this chapter, I describe the research method for the empirical study. This chapter includes: 5.1. Content analysis and the analyzing process, including the choice of mathematics textbooks; 5.2. Analytical criteria and categories, including the choice of mathematics textbooks; 5.3. Quality in this study; 5.4. The clarification of some terms used when analyzing the chosen textbook.

5.1 Content analysis and the analyzing process

The empirical approach in this research is inferred from content analysis (Cohen, Manion, & Morrison, 2007; Krippendorff, 2004; Weber, 1990). Referring to Weber (1990), Cohen et al. (2007) point out that content analysis was originally derived from analysis of mass media and public speeches, but the use of content analysis has spread to examination of any form of communicative material, both structured and unstructured. They claim that content analysis as a research tool can be applied for any written material, from documents to interview transcriptions, from media products to personal interviews. They criticize the fact that the term content analysis has been used quite carelessly. Cohen et al. (2007) agree that content analysis defines a systematic set of procedures for rigorous analysis, examination and verification of the contents of written data. It involves procedures like coding, categorizing, comparing and concluding (Cohen et al., 2007).

Quantitative content analysis often deals with texts through counting textual elements but may miss syntactical and semantic information embedded in the texts according to Cohen et al. (2007). Hsieh and Shannon (2005) state that *qualitative content analysis* involves not only counting words from texts but also attempting to explore the meanings underlying the texts and to generate theory. The process of qualitative content analysis uses inductive reasoning. They mention that the conventional approach is that researchers code categories derived directly and inductively from raw data. Another approach is that researchers initially code text using a theory or relevant research findings and then allow themes to emerge from the data in order to validate a conceptual framework or theory (Hsieh & Shannon, 2005). Summative content analysts start with counting words and then finding the latent meanings and themes in an inductive manner (Hsieh & Shannon, 2005). Krippendorff (2004) points out that “qualitative researchers tend to apply criteria other than reliability and validity in accepting research results” (p. 88). There are many alternative criteria for qualitative content analysis such as trustworthiness, credibility, transferability, embodiment, accountability, reflexivity, and emancipator aims (Denzin & Lincoln, 2000). However, the content analysis methodology in my study is not an attempt to do quantitative analysis through counting coded words, but instead it is closer to qualitative approaches to text interpretation that is interpretive and shares certain characters with the content analysis approach in terms of coding algebraic contents in textbooks. The analog characteristics are for example a close reading of relatively small amounts of texts and interpreting given texts into analytical narratives (Krippendorff, 2004). The analysis consists of my interpretation of mathematical texts in the textbooks using criteria derived from previous research on mathematics textbooks.

The empirical material for this study consists of mathematics textbooks. I use the term *algebra content* to refer to mathematics as a discipline. *Algebra content topics* are polynomials, distributive law, factorization, square rule, the difference-of-two squares, quadratic formula and so on. These topics are the titles of different content parts in the textbooks. Mathematical *textual presentations* refer to explanations of mathematics concepts and rules in the textbooks. The empirical analysis is done in four rounds of analyses as described in the following sections.

The first round of analyses

To explore the embedded pedagogical content knowledge of teaching algebra, I chose to start by investigating what algebra content topics that are presented and if the topic of factorization is present in eight textbooks. The eight mathematics textbooks available in the university library collection were selected with one book from each textbook series. By looking at all algebra topics, including both quadratic equations and quadratic functions, which are presented in these eight textbooks, I have obtained an overall view of which algebra content topics related to solving quadratic equations and functions that are presented in textbooks at the course B level. The *result* shows that all algebra content topics related to solving quadratic equations in each textbook are the same and factorization for the standard quadratic polynomial like $2x^2 + x - 6 = (2x - 3)(x - 2)$ is absent in every textbook. The algebra content topics are presented in different order in the eight books. Some books have the chapter on quadratic equations before the chapter on quadratic functions and some the other way around.

The second round of analyses

Based on the result from the investigation of the eight textbooks, three of the eight books were chosen for a more detailed analysis. The selection of the three mathematics textbooks was based on the demands of the textbook market in Sweden in 2008 when my analyzing work started. Textbook market investigation was carried out through e-mail contacts with three large Swedish publishers. The three selected mathematics textbooks for mathematics course B were *Matematik 4000* (the Blue book) (Alfredsson et al., 2007), used by students who study in the science and technology programs and who will continue to study advanced mathematics later; *Matematik 3000* (Björk et al., 2000), used by students who study in the social and vocational programs and who may continue with further studies of mathematics in the future; and *Exponent B* (the red book) (Gennow et al., 2005b), used by students who study in the science and technology programs and who will continue to study advanced mathematics later. Another reason to choose these three books is that two of them have rich exercise alternatives and are designed for the above-average level mathematics students and one of them is designed for the average level mathematics students.

To carry out detailed analyses of the three selected mathematics textbooks, I have used a CK-PCK (Mishra & Koehler, 2008; Shulman, 1986b) framework, combined with different criteria derived from the previous studies on textbook research as analytical tools (Brändström, 2005; Pepin et al., 2001; Van Dormolen, 1986). I have analyzed the first textbook by describing what is presented in the related sections of algebra, considering textual presentations, examples and exercises in detail page by page; then compared the first book with the second and third selected mathematics textbooks. The preliminary empirical data of the content analysis has resulted in three descriptive analytical texts of the three selected textbooks.

The third round of analyses

The detailed analyses of the three textbooks did not alone provide enough material to answer my research questions. I decided to go back to a general comparison again among the eight selected textbooks and also widen the range. At this time, I went through all the published mathematics textbooks for mathematics course B published by four big Swedish publishers, and added four more selected textbooks to the eight previously chosen. The selection was intended to cover various mathematics levels to the highest degree possible. In total, there were now twelve selected textbooks (see Appendix 1) for the third round of analyses. The investigation of the twelve books covers two mathematics areas: algebra and functions. The analyzed objects in this round of analyses are content table and *content elements*[9] in every textbook. All the content elements, including mathematical content and pedagogical activities, are coded in short but clear phrases or words in order to be easily compared in every table (see Appendix 2). Twelve tables (one table for each investigated textbook) are used to compare the analyzed objects.

The aim of this round of analyses was to see if there are links between quadratic equations and functions by investigating the order of the presentations of quadratic equations and functions. In the history of algebra, function came after quadratic equations (Kvasz, 2006), but in Swedish textbooks, function is not treated in the area of algebra (Jakobsson-Åhl, 2006). If functions are presented before quadratic equations, there could be some pedagogical intention to do so. Could that influence the presentation of quadratic equations? This would relate to the first research question: What mathematics do Swedish upper secondary textbooks reflect in its presentations of quadratic equations? Guided by this idea, I investigated the logic of presentation orders and organization of the textbooks, concerning both mathematics as a discipline (CK) and pedagogical content knowledge (PCK) aspects.

In this round of analyses, I focused on the order of the presentations of the algebra content and content complexity as well as content connections in every book. During the work of analyzing, I have again reduced the number of textbooks from twelve to eight, a reduction due to the content complexity of mathematics in the books. I excluded another four textbooks out of the twelve because they either contained less mathematics topics or they were aimed for students who will not continue to study advanced mathematics later in upper secondary school. From the limited data, a pattern of essential content elements emerged for each textbook.

The *result* coheres with the result derived from the first round of analyses and it indicates that all the mathematical content in these books are similar since they have the same content topics, but that the textbooks present the content in different order. *Five out of the eight textbooks present quadratic equations before quadratic functions*. Three of the eight textbooks present quadratic functions before quadratic equations, but only one of those three books explicitly presents the relationship between quadratic function graphs and solving quadratic equations. Without considering the different order of presenting quadratic equations and functions, I excluded four books out of the eight ones since they are similar in content structures. The four textbooks differing mostly from each other are left for the next round of analyses. They are: *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007), *Exponent B* (the red book) (Gennow et al., 2005b), *Origo B* (Szabo et al., 2008) and *Räkna med Vux B* (Danielsson et al., 2002). Among these four chosen books, I needed to select one of them for a deep analysis. Based on the result of the five books presenting quadratic equations before quadratic functions, I decided to choose either *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007) or *Exponent B* (the red book) (Gennow et al., 2005b) for the fourth round of analyses.

[9] Content elements in a mathematics textbook refer to all parts in every unit and come after every chapter. They are often parts like: new mathematical content presentation, examples, exercises at three levels after the presentation, historical background of the related mathematical content, chapter review, mathematics activities before and after a chapter or unit, test, and a mixed exercise set et cetera.

The fourth round of analyses

Starting with the two textbooks selected from the third round analysis, I have been in contact via e-mail and telephone interviews with the related Swedish publishers. According to their information, *Matematik 4000* was regarded as the most used mathematics textbook in Sweden. at the time when I interviewed them. Therefore I decided to focus on *Matematik 4000* (the Blue book) (Alfredsson et al., 2007) for the more detailed content analysis. Grounding data of *Matematik 4000* (the Blue book), including analysis of algebra content textual presentations and of exercises and activities, was generated in the second round of analyses. Based on these, the textbook *Matematik 4000 B* (the Blue book) has been analyzed three times with different focuses:

1. Detailed analysis of the content elements including content textual presentations and exercises as well as activities and tests in the sections of algebra and functions. This analysis has been done page by page in a descriptive way. In total 85 pages of the textbook have been analyzed. The analytical criteria used in the analysis are based on the criteria derived from previous studies in the field of textbook research (Brändstöm, 2005; Pepin et al., 2001; Van Dormolen, 1986). Algebra concepts, definitions and operational rules presented in Chapter 3 were used as references in the analysis. When examining the exercises and activities in the textbooks, I solved them step by step the way a student is expected to, and thus I investigated the mathematical complexity of the tasks in relation to concepts and procedures as well as demands on the student. The aim of this analysis was to find the subject content knowledge of algebra (CK) presented in the textbook.
2. Analysis of the same *content elements* with focus on connections between content elements presented in the textbook and algebra history presented in Chapter 3.1, with the aim of finding historical connection between algebra content in the textbook and algebra history.
3. Analysis of the organization of the content elements in search of an embedded teaching trajectory. One result of the previous rounds of analyses revealed a cumulative sequence for organizing the algebra content related to quadratic equations since each part is a development of the previous part and linked to another logically by geometrical and algebra representations. To explore the embedded *teaching trajectory*, I decided to limit the *content elements* to the sections of quadratic equations on 48 pages with focus on geometry models and algebra representations as well as mathematics applications. The three important organizing categories used in this analysis are: 1. *Mathematical content* (examining algebra content from a mathematics as a discipline point of view (Van Dormolen, 1986) and an applied mathematics point of view (De Lange, 1996); 2. *Pedagogical activities* (what has been done in order to present algebra content and make it comprehensible); 3. *Other notes* (mathematical consistency and clarity through language use and definitions as well as mathematics correctness). The categories used in this analysis will be presented in the next part.

5.2 Analytical criteria

The overall analytical framework used in the empirical study of analyzing algebra content in mathematics textbooks is CK-PCK (Mishra & Koehler, 2008; Shulman, 1986b) as presented in Chapter 2.2. The content analysis involves algebra content knowledge as subject matter content knowledge from mathematics as a discipline aspect. When examining the algebra content textual presentations, examples, exercises and activities, I have applied combined analytic criteria from a previous study in this area (Pepin et al., 2001). These criteria are derived from a combination of the classification aspects used by Van Dormolen (1986) and Schmidt et al. (1997) in a cross-nation textbook analysis study. Pepin et al. (2001) consider it important to investigate intended views of the nature of mathematics projected in textbooks. Referring to Van Dormolen (1986), they mean that an analyst might look for the following aspects when analyzing mathematics textbooks:

> […] a theoretical aspect (theorems, definitions, axioms); an algorithmic aspect (explicitly how to do…); a logical aspect (rules about how we are and are not allowed to handle theory); a methodological aspect (how to do…more heuristically, for example how to use mathematical induction); a communicative aspect (conventions, or how to write down an argument, for example). Schmidt et al. on the other hand, classify an understanding of the content in terms of its: topic complexity (which topics; when; which emphasized; with what conceptual demands); developmental complexity (ways of sequencing and developing topics across lessons and across the whole curriculum for example, focused and concentrated or a spiral of revisiting topics); cognitive complexity (the pedagogical intention for the topic i.e. what you want the students to do as a result of having learnt the topic). (Pepin et al., 2001, p. 4)

Van Dormolen's classification seems to be intended to explore the nature of mathematics representing a teaching perspective while Schmidt et al. focus their classification on a learning perspective. Based on their classification and the CK-PCK overall framework, the following criteria have been considered for analyzing algebra content textual presentations[10], examples, and different activities in my study:

1. *Consistency and clearness of Mathematical content:* A mathematical text should be consistent and clear to the reader. "There must be no errors, either of computation or of logic. Proofs might be incomplete, but not false. Conventions must be used consistently. […] the content must be clear to the intended reader." (Van Dormolen, 1986, p. 151).
2. *Mathematical theoretical aspects*: This criterion concerns knowledge elements such as mathematical theorems, rules, definitions, methods and conventions. Such mathematical knowledge is called "kernels" (Van Dormolen, 1986, p. 146). By means of this criterion, I investigate what and how mathematical concepts or terms, definitions, methods, rules and theorems as well as examples are defined and explained in algebra content textual presentations foregoing every exercise set in the textbook. In such a way, I answer the two sub-questions of the second research question: 2a) How is mathematical content presented or explained? 2b) What are the character and function of the presented examples and exercises?
3. *Mathematical content development and connections*: This criterion is based on the classification of Schmidt et al. (1997). By means of this criterion, I investigate how mathematical content topics relate to each other in the chapter of algebra. The aim is to explore the embedded teaching trajectory related to quadratic equations. I seek if "there is a logical progression in the sense that students need to know some kernels in order to be able to understand others" in the mathematics text in the textbook (Van Dormolen, 1986, p. 152). In other words, I consider if the textbook presents the origins and extensions of

[10] An algebra content textual presentation refers to a core part of every unit in a mathematics textbook. In this core part, new mathematics concepts or rules or mathematical procedures are introduced in words and mathematical formula. After the content textual presentation, some related examples with procedural steps and answers are often illustrated in order to make readers understand what has just been presented.

core concepts by following pedagogical ideas of developing a progression from a basic to a more abstract level. If so, what is the final goal of the progression?

4. *Mathematical representations and applications*: Aspects of mathematical representations and applications are taken into account since they often reflect different views. A formalistic view regards mathematics as a set of concepts, rules, theorems and structures. Mathematics applications are often regarded as informal view. In an informal view students are encouraged to engage in activities like generalizing, classifying, formalizing, ordering, abstracting, exploring patterns and so on, and new ideas are encouraged (De Lange, 1996; Freudenthal, 1991; Goldin, 2008; Pepin et al., 2001; Van Dormolen, 1986; Vergnaud, 1987). With this criterion, I search the character and functions of the presented examples and exercises in the textbook, and look for the embedded pedagogical content knowledge in order to answer the second research question: What aspects of pedagogical content knowledge can be traced in the way a Swedish upper secondary school textbook presents the algebra content of quadratic equations?

5. *Language use*: In which way are mathematical theorems, definitions, and rules explained and illustrated: formally in a mathematical language or pedagogically in combination with everyday language, in order to make sense for a student reader? For example, factorization procedure is explained by special expressions "inversely" and "breaking out" (Alfredsson et al., 2007, p. 21) in the textbook. Such expressions are close to everyday language. This criterion is not applied independently but is involved in criterion one and two above.

To analyze different kinds of mathematics exercises, activities and problems as well as tests in the textbook, I have applied some ideas from a framework used in a previous study and which has been mentioned in Chapter 4.1.3 (Brändström, 2005). To involve exercise analysis in algebra content analysis is to see the wholeness of the analysis in regard to the aims of this research: investigating the algebra content as subject content knowledge and exploring the embedded teaching trajectory for teaching quadratic equations. The following categories are taken into account in the analysis of different exercises and activities.

A. *Routine exercises* refer to the kind of exercises that require students to use newly presented mathematical concepts, rules or algorithmic procedures illustrated in examples, in order to get familiar with the content. This kind of exercises is often at a basic level and requires simple and similar operations or reasoning to those just presented. For example, when the difference-of-two squares formula and square rules (as algebra content) have been presented in the textbook, one of the offered exercises is (Exercise 1145): "Extend with the help of the difference-of-two squares formula: a) $(x+3)(x-3)\ldots$" (Alfredsson et al., 2007, p. 19). The operating procedure only consists of one step for this task and the formula has just been presented in this part.

B. *Exercises that require students to evaluate, analyze and reason mathematically* instead of merely computing mechanically (Brändström, 2005). Such exercises intend to encourage students to understand the integration of mathematics concepts and procedures (Hiebert & Carpenter, 2007; Hiebert & Lefevre, 1986). For example, one of the exercises after presenting quadratic formulas in the section of quadratic equations in the book is Exercise 1228 (Alfredsson et al., 2007, p. 30):

> Louis and Nille want to solve the equation $x^2 + x - 2 = 0$ with quadratic formula.
>
> Lois: "One of the coefficients of *x* is missing, *p* is 0."
> Nille: "We have one *x*, *p* is 1."
> a) Who is right?
> b) Solve the equation.

This exercise concerns the understanding of number sense and the concept of the coefficients of the variable *x*. Many students might not be able to tell the difference between x and $1x$. If *p* is 0, what is implied for *x*? Realizing $0x = 0$ is different from $1x = x$. The exercise also requires students to be able to judge what is right or wrong and tell the difference. The equation can be solved using any of three previously learned algebraic solving methods or a graphical method. This exercise is related to both conceptual understanding of the structure of quadratic equations and to computation procedures.

C. *Exercises that require students to understand the structure of quadratic equations*. This kind of exercises is often introduced by giving some information for a quadratic equation or polynomial while leaving something unknown, which requires the student to find the unknown value. One example is Exercise 1232 (Alfredsson et al., 2007, p. 30): "For which value of *a* does the equation have no real roots? $a) x^2 + a = 4 \quad b) x^2 - 12x + a = 0$".

   This exercise requires the student to find that the two equations could be rewritten in the form of $(x+?)^2 = ?$ and equivalent relationship between them. The principle is that the perfect square trinomial is always positive. When the value of a perfect square trinomial is negative, the equation lacks real roots. Therefore the discriminant[11] must be positive. This exercise requires a mathematical analysis. The procedure is carried out by using either the method of completing the square or square root or the quadratic formula. The exercise provides students with an opportunity to analyze and understand the relationship between a quadratic equation's roots of real numbers and its coefficients and constants from an algebraic structure aspect.

D. *Exercises that are related to real world contexts*. Such exercises are often word problems (or called real world problems) and the pedagogical reason of using them is to bring reality into the mathematics classroom, to create occasions for learning and practicing the different aspects of applied problem solving without the practical contact with the real world situation (Chapman, 2006). They reflect the view of mathematics applications in real-life situations as mentioned in point 4 above (De Lange, 1996; Freudenthal, 1991; Goldin, 2008; Pepin et al., 2001; Van Dormolen, 1986; Vergnaud, 1987). Exercise 1251 serves as an example (Alfredsson et al., 2007, p. 36).

   > In a 2000-year-old Chinese writing "Nine chapters calculation art" ("Nine chapters arithmetic"), we find the following problem: "In the middle of a square lake with its side at *s* meters, there is a reed growing *h* meters over the surface of the lake. If the reed is dragged toward the side of the lake, it reaches exactly the surface of the lake. The depth of the lake is *d* meters. Show that:
   > $$d = \frac{s^2}{8h} - \frac{h}{2}$$

   This problem requires at least five solving steps: 1. Reading the word problem and understand it. 2. Drawing a picture of the lake and reed, marked with given data. 3. Drawing a right-angled triangle derived from the interpretation of the first two steps. 4. Setting up a quadratic equation with the help of the third step and Pythagoras' theorem. 5. Algorithmic calculating of the quadratic equations by symbols only and carrying out the proof.

[11] Discriminant is the expression of $b^2 - 4ac$ to a quadratic equation. It is part of the quadratic formula and can be used to analyze the roots situation of a quadratic equation (Great Source Education Group, 2000).

E. *Mathematical proofs* are often purely symbolic computations according to rigorous algorithmic procedures. Generalizing and inductive reasoning are needed in such exercises or activities. As an example one can look at exercise 1234 (Alfredsson et al., 2007, p. 30):

> In other countries, "abc formula" is used instead of our "the pq-formula." Show that the equation $ax^2 + bx + c = 0$ has the solutions:
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

F. *Different possible methods for solving quadratic equations* are considered in the exercises related to solving quadratic equations since this is the essential algebra content. As an analyst, I go through all the exercises and solve quadratic equations provided by the textbook with different solving methods in order to see how many potential solving methods that can be used and which one is efficient. To judge if the solving method is efficient or not, I examine how many procedural steps need to be taken to solve a quadratic equation. According to a previous study (Bossé & Nandakumar, 2005), the solving method depends on the type of quadratic equations that are decided by the coefficients and constants of a quadratic equation. To examine the relation between solving methods and different kinds of quadratic equations, I categorize them according to equation types; solving methods and the number domain of coefficients and roots (see Table 2). After all the related solving methods have been presented in the textbook, there is an exercise set providing 14 exercises and containing in total 28 quadratic equations in which students are to practice the recently presented solving methods. The following table shows my analysis of 19 out of 28 quadratic equations for seeking the most efficient solving methods according to my categories. Nine out of the 28 equations are not included here since they have either constants or coefficients expressed by pure algebraic symbols instead of numbers, for example $ax^2 + bx + c = 0$.

Table 2

The selected 19 out of 28 quadratic equations applied for practicing four solving methods[12] (Alfredsson et al., 2007, p. 30)

Tasks	SQR[13]	CSQ[14]	NF[15]	PQ[16]	Number of Methods	Roots
$x^2-6x+5=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$x^2+6x+5=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$x^2+4x-12=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$5x^2-15x+10=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$-4x^2+44x-96=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$x^2-4x+1=0$	No	Yes	No	Yes	2	Real
$2x^2+24x+74=0$	No	Yes	No	Yes	2	Irrational
$8z^2-8z+2=0$	Yes	Yes	Y(NF)	Yes	4	Double roots, rational
$A(A+10)=39$	No	Yes	Y(NF) easiest	Yes	3	Integers
$x^2+x-2=0$	No	Yes	Y(NF) easiest	Yes	3	Integers
$2x^2+2=12$	Yes	No	No	Yes	2	Real
$10y-y^2=5$	No	Yes (best)	No	Yes	2	Real
$14x-x^2=0$	No	Yes	NF direct	Yes	3	Integers
$x^2-3x+1=0$	No	Yes	No	Yes easiest	2	Real
$\frac{3}{5}x^2-2x-5=0$	No	Yes	Y (difficult but effective)	Yes easiest	3	Rational
$0,4x^2+1,2x-1=0$	No	Yes	No	Yes easiest	2	Real
$-6n^2-5n+1=0$	No	Yes (more difficult)	Y (difficult but effective)	Yes easiest	3	Rational
$z^2+12z+1=0$	No	Yes (easiest)	No	Yes	2	Real
$2x^2-8x-10=0$	No	Yes	Y (easiest)	Yes	3	Integers
Total: 19 equations	2/19	17/19	12/19	19/19		

[12] Four solving methods are presented in these two sections in the book. They refer to the square root method; null-factor law method (or factorization); completing the square method and a quadratic formula (or *the pq-formula*).
[13] SQR represents the square root method for solving quadratic equations.
[14] CSQ represents the completing the square method for solving quadratic equations.
[15] NF represents the null-factor law method for solving quadratic equations. It is actually a method of factorization.
[16] PQ represents the quadratic formula method for solving quadratic equations and is often called *the pq-formula* at school.

5.3 Conclusion of the analyzing process

The first round of analyses of the eight selected textbooks answers the research question: What algebra content related quadratic equations are presented in the investigated mathematics textbooks? What is the most emphasized solving method for solving quadratic equations presented in the textbooks? The second round of analyses contains three descriptive and comparative analyses in detail and answers the research question on how factorization is presented in the textbooks. The derived analytical data set up a base for the in-depth analysis later. The third round of analyses explores the order and connections in organizing mathematics content related to quadratic equations and functions. The result of this process has helped to narrow down the number of selected textbooks. In short, the first three rounds of analyses aimed at answering the first research question and setting up a basis for the fourth round of analyses regarding the embedded teaching trajectory.

Based on the results derived from the first three rounds of analyses, the fourth round of analyses was narrowed down to one selected textbook to be analyzed in more detail. The fourth round of analysis is carried out with three different focuses in order to answer the research questions two and three: What aspects of pedagogical content knowledge can be traced in the way a Swedish upper secondary school textbook presents the algebra content of quadratic equations? Is there any embedded teaching trajectory (sequence) built in the presentations of quadratic equations in the textbook? How can that embedded teaching trajectory be described? To carry out the second and fourth rounds of analyses, I have used the above described five analytical criteria (1-5) for analyzing algebra content textual presentations and mathematical examples; and six categories (A-F) for analyzing exercises and activities. The answers to the last two research questions have been yielded after the last detailed analysis of the textbook *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007).

5.4 Quality in this study

The empirical objects in this content analysis are mathematical texts presented in textbooks. Differing from other kinds of texts, mathematical texts are often short and instructional. In the texts, there are mathematical terms and rules as well as examples that serve as introductions to new mathematical content like instructions used for the purpose of teaching. In analyzing, I do not intend to code words for quantitative analyses, but instead I seek the links between mathematics contents and so I interpret mathematical texts and exercises according to what mathematics content they represent or infer, considering both mathematics as a discipline and pedagogical perspectives. Therefore, I code mathematical content presented in the investigated textbooks into simple phrases in order to find the links between the contents among the textbooks (see Appendix 1). My interpretations of mathematics content are described in words and mathematics symbols, representations and geometrical figures. Such interpretations are written in a form of descriptive texts using the described analytical criteria and categories. Thus, the study results yield descriptive texts, which describe what I have interpreted and coded from the textbooks, both at an overall level and detailed level, in a narrative way.

Lester and Lambdin (1998) provide some criteria for evaluating quality of mathematics educational research: worthwhileness, coherence, ethics, credibility, and other qualities.

Worthwhileness mainly indicates:

> The study generates good research questions, the study contributes to the development of rich theories of mathematics teaching and learning, the study is clearly situated in the existing body of research on the question under investigation, and the study informs or improves mathematics education practice (Lester & Lambdin, 1998, p. 420).

Coherence has to do with validity (Johansson, 2006). Simply, it means that research questions and methods should match each other. It is about "whether a research design will generate evidence that is appropriate for the question being asked" (Lester & Lambdin, 1998, p. 421).

Ethics concerns "(a) the manner in which research has been conducted in relation to the research subjects (often students or teachers), and (b) acknowledgement of contributions of others" (Lester & Lambdin, 1998, p. 422).

Credibility is a synonym of trustworthy. Research findings should be based on or grounded in data or evidence. The claims and conclusions drawn should be justified and reasonable without purely relying on rhetoric. It should be possible to verify and refute the arguments and interpretations written in the research report (Lester & Lambdin, 1998).

Other qualities of good research reports have insubstantial characters such as: lucid, clear, well organized, concise and direct. By referring to Kilpatrick (1993) and Sierpinska (1993), Lester and Lambin (1998) have mentioned originality for good quality research which indicates that such research provides a new technique of analysis and a new interpretation for old data.

With regard to *worthwhileness*, this study on mathematics textbook analyses attempts to contribute to the mathematics community, in particular the areas of teaching algebra and the topic of solving quadratic equations and even textbook writing. As mentioned in Chapter 4.2.1, research on quadratic equations in the area of school algebra has not received much attention yet (Kieran, 2007). Studies on analyzing mathematics textbooks with a CK-PCK focus are also few. This study tries to combine both algebra content as CK aspect and pedagogical intentions as PCK aspect in the analyses. Considering *originality*, it is the researcher's ambition to intend to bring these two traditional fields into the field of textbook analysis.

With regard to *coherence*, this study has followed its general aims to study algebra content and pedagogical intentions presented and implied in textbooks. Every round of analyses tries to seek the answers to the research questions. The design of my study is based on the research questions.

With regard to *ethics*, this study uses all the textbooks available from the libraries of the university where the researcher works and they are all published in Sweden and distributed on an open market. As a researcher, I have informed the related publishers and sent them my acknowledgement of their contributions of market information.

Regarding *credibility*, I follow the principle 'let the data speak' when I am analyzing the textbooks. The findings from this study are derived after four rounds of analyses of a number of the textbooks. Deeper analyses of one book have been carried out and seven descriptive analyzing texts have been written until answers to the research questions have emerged from the analyzed data. To some extent, the analyses are limited since my interpretations of algebra content and pedagogical intentions are based on my own experiences as an upper secondary school mathematics teacher and college student in the subject of algebra. With assistance from my supervisors and colleagues who have read and discussed the analyzed texts, I have revised my analysis many times in order to make the interpretations as trustworthy as possible (Kilpatrick, 1993). The theoretical aspects and analytical categories applied to my study are derived from previous research studies in order to be reliable. The mathematical links with abstract algebra in Chapter 3.3 have been discussed with a university lecturer of mathematics with specialty in algebra in order to be mathematically correct.

Considering *cumulative aspect* of this study, I have carried out four rounds of analyses including twelve textbooks. The first round of analyses has found the answers to research questions 1a), 1c) and 1d) within the eight mathematics textbooks for mathematics B course. The second round of the analyses found more detailed answers to research question 1d) and the part of question 2c) related to embedded teaching trajectory within the three selected mathematics textbooks. When the evidence was not enough to answer all the sub-questions to the first research question, I decided to carry out the third round of the analyses, including four more textbooks, in order to find an answer to research question 1b) and to look for the final goal of the embedded teaching trajectory. The third round of analyses helped to narrow down the number of selected textbooks. Based on the previous results derived from the first three rounds of analyses, I narrowed down the number of textbooks to one textbook for the detailed analysis of algebra content and exploration of embedded pedagogical intentions and teaching trajectory. In the fourth round of analyses, the same textbook content has been analyzed with three focuses in order to find answers to the second research question. After the fourth round of the analyses, the evidence generated is enough to answer research questions 2. In such a way, this study contains a cumulative process in the analyses.

The number of the textbooks chosen changes depending on the need for the analyses and the change follows this order: 8 → 3 (chosen from the first 8 books) → 12 (after adding 4 more books to the first 8 ones) → another 8 (after deselecting 4 books from the 12 ones) → 4 (after deselecting another 4 books from the 8 books this time) → 2 (after deselecting 2 books from the 4 books) → 1 (after deselecting 1 book from the 2 books).

To explain this procedure, I first investigated eight mathematics textbooks for the B course in order to obtain an overall view of algebra content related to quadratic equations. In the second round of analyses, I chose three out of the eight textbooks to make detailed analyses. In the third round of analyses, I added four more books to the first eight textbooks and then the number of the investigated textbooks became twelve. After a selection among the twelve textbooks, I deselected four books, leaving another eight textbooks left for the third round analysis. These eight textbooks, however, are not exactly the same as the first eight books from the first round of analyses. After the third round of analyses, I deselected another four textbooks from the second set of eight. I then chose two of the four textbooks for the fourth round of analyses. In the fourth round of analyses, I finally chose one out of the two books to analyze deeply.

5.5 Clarification of some terms in the fourth round of analyses

Some terms, such as mathematical "exercise" and "task," "real world problem," "word problem" etc, used in this analysis need to be clarified in order to assist the readers. It should be noted that these notions are defined within the analysis and they can not be used in a general sense. The purpose of giving meanings of these terms is not to make epistemological concepts but to provide working definitions for this analysis.

A *unit* is a component of a section in the textbook. A unit consists of three parts: textual presentation of mathematical content, examples illustrating the presented content in the textual presentation part, and exercises offered for students to practice the newly presented mathematical content.

Textural presentation of mathematical content refers to the first part of every unit before given exercises. It consists first of a presentation of mathematical content related to a mathematical topic in every unit and then illustrated examples with given answers consisting of all procedural steps. This part includes statements of mathematical concepts, terms or operational rules or methods and illustrated examples. Such a presentation is equivalent to the Swedish word "genomgång" which means to work out or go through mathematical contents in a pedagogical way.

Exercises refer to all the mathematical tasks provided in the book after the *textual presentation of mathematical contents* in every unit. They are divided into *a*, *b* and *c* levels. *Sub-exercises* are a number of mathematics tasks or questions included in one exercise. An exercise part (or sometimes called exercise set) contains the three levels of exercises, including routine exercises or decontextual exercises and contextual ones. Sometimes the exercises are word problems with or without contexts.

Tasks refer to mathematics exercises offered in tests at the end of every chapter. Tasks and exercises are similar, but a task in this analysis is based on the interpretation from Haladyna (1997) that a task can be an instruction or question requiring a student's response under certain conditions and specific scoring rules.

Activities refer to those mathematical activities provided at the beginning and at the end of a section with the pedagogical purpose of organizing students to work in groups solving a number of mathematical exercises or problems.

Tests refer to the mathematics tasks provided at the end of every chapter. They are similar to mathematical national tests at this level.

Problems for everybody refer to mathematics problems provided after tests at the end of every chapter. Those are all word problems including both contextual and decontextual ones.

Boesen (2006) has pointed out the multi-meanings of the term "problem":

> The term "problem" is used with many different meanings, and can in principle mean everything from a dressed up exercise to front-line research. […] Whether a task constitutes a problem or not depends both on the solver and the task. What might be a problem for one student might not be so for another. (Boesen, 2006, p. 4)

His interpretation is based on the discussion from Schoenfeld (1985):

> Being a "problem" is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a

> problem for that person. The word problem is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one. […] If one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem (as cited in Boesen, 2006, p. 74).

In this analysis, the objective is not to study students, and in that sense it is difficult to judge if a mathematical task is a problem or an exercise. Therefore, the analysis tries to follow the terms given in the textbook. Exercises after every presentation part are called *exercises*. Different activities offered after every section is called *activity*, though it consists of a number of exercises. Tests containing a number of tasks offered after every chapter are called *tests*. *Problems* offered after tests are called *problems for everybody*. In my analysis, however, I regard exercises that are given in words and have either daily or mathematical contexts as word problems without following Schoenfeld's interpretation of problem (1985).

Word problems refer to mathematical problems that are presented in verbal form combined with mathematical symbolic and pictorial forms. In mathematics classroom research, real world problems are viewed as word problems related to daily life contexts:

> Word problems can be used as a basis for application and a basis of integrating the real world in mathematics education. They can provide practice with real life problem situation, motivate students to understand the importance of mathematics concepts and help students to develop their creative, critical and problem solving abilities (Chapman, 2006, p. 212).

Real-world problems are understood, by some scholars, as mathematical problems involved in "the rest of the world outside mathematics, i.e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us" (Blum & Niss, 1991, p. 37).

The pedagogical intentions of using real world problems are discussed by some scholars. By referring to Verschaffel (2002), Chapman (2006) points out that the pedagogical goal of using word problems is to bring reality into the mathematics classroom, to create occasions for learning and practicing the different aspects of applied problem solving without the practical contact with the real world situation. Similarly, according to Boaler (1994) and referred to by Chapman (2006), the purposes of using such problems are to "provide students with a familiar metaphor, to motivate and interest students, and to enhance the transfer of mathematical learning through a demonstration of links between school mathematics examples and real world problems" (p.212).

This analysis uses the term *real world problems* to represent mathematical problems, exercises or tasks related to real-life situations and other school subjects than mathematics. The real world problems belong to word problems. However, *word problems* can also refer to exercises within a mathematical context if they are expressed in words.

Decontextual exercises refer to mathematics exercises isolated from any context except for pure mathematics and presented only in symbolic form, with short instructions to guide students in doing the exercises. For example, "Simplify," "Factorize," "Solve the equations" etc. Such mathematical exercises are routine ones at the A-level and often placed in the beginning of every exercise part, using procedures that have been illustrated in the previous presentation of new mathematical contents and examples. Routine exercises can be carried out easily using similar operational procedures or algorithmic steps as the ones illustrated in the examples. In contrast to decontextual ones, *contextual exercises* refer to exercises that are described and constructed within a certain mathematical or daily life related context in mostly words and some mathematical expressions and sometimes pictures as well. The contextual exercises should be regarded as the same as the word problems mentioned before. They have the same meaning in this thesis.

The expressions “easy” and “difficult” regarding mathematics exercises or problems contain the following meanings: *easy* mathematics exercises or problems refer to those routine ones mentioned above. The solution procedures are often available in the previous presentation parts. Students can solve them by looking for and copying or imitating procedures earlier in the same textbook section without any deeper understanding (Boesen, 2006). Those can also be word problems requiring one or two solving steps.

Difficult mathematics exercises or problems refer to non-routine ones which are rare or not found in the previous presentation parts or require complicated procedures, often in more than three steps. They may require such cognitive activities as judging, finding or correcting mistakes, reasoning, generalizing, proving etc. They can be decontextual problems or abstract problems that focus on pure algebraic symbolic operations requiring many complicated steps. On the other hand, they can also be contextual problems requiring fewer operational steps but a phase of interpretation into algebraic expressions.

Some mathematics problems may require students’ *creative* thinking which is related to deep, flexible knowledge. It is often difficult for students to find out solving methods or paths for such mathematics problems directly since they are not obviously related to what has just been presented in the book, at the same time as more than one solving approach can be used. These creative mathematics problems require that students really understand mathematical meanings conveyed by the problems and thereafter find a relevant solving approach without being limited by the actual mathematical knowledge presented. The term creative thinking in my analysis of exercises has been borrowed from the concept of creative mathematically founded reasoning (CR in short) in Boesen’s study (2006) without sharing the exact meaning of CR.

The expressions *abstract* and *concrete* in this analysis have their literal meanings. “*Abstract*” refers to a mathematical problem or exercise that is expressed or instructed only through mathematical symbols without being related to any context while “*concrete*” is the antonym to abstract.

6. Results

In this chapter, I will present study results of the four rounds of analyses of mathematics textbooks according to the order of the research questions asked in the first chapter and concerning both mathematics as a discipline and pedagogical content knowledge. The whole chapter consists of three parts in which the two research questions will be answered followed by a summary.

6.1 What mathematics do Swedish upper secondary mathematics textbooks reflect in the presentations of quadratic equations?

The first three rounds of analyses aim at answering the first research question: What mathematics do Swedish upper secondary mathematics textbooks reflect in its presentations of quadratic equations?

a) What algebra content related to quadratic equations is presented in the investigated textbooks?
b) In which order is quadratic equations and functions presented and do they have connections to each other?
c) What is the most emphasized solving method for solving quadratic equations presented in the investigated mathematics textbooks?
d) How is factorization presented in the investigated textbook?

The results derived from the first round of analyses have answered the research question 1a), 1c), and partly 1d). The first round analysis shows that *the algebra content relating to quadratic equations and quadratic functions are similar in all the eight investigated mathematics textbooks since they have the same algebra content topics*. The same result was obtained in the second and third rounds of analyses. Algebra content elements related to quadratic equations are: simplifying polynomials, binomial multiplications, distributive law, the difference-of-two squares formula, square rules, solving simple quadratic equations with the square root method, null-factor law (factorization), solving general quadratic equations with the completing the square method and the quadratic formula. Among these core content elements, *solving quadratic equations with the quadratic formula is always presented at the end*. The embedded pedagogical content knowledge is revealed when these core algebra content elements make up the necessary pre-knowledge for a student when learning quadratic equations and their solving methods. Presentation of different kinds of quadratic equations is restrained within simple and complete quadratics of the type $x^2 + px + q = 0, (p \neq 0, q \neq 0)$ in all the books. General quadratic equations of the type of $ax^2 + bx + c = 0, a \neq 0$ are absent in six books. The transformation from general quadratic equations to complete quadratic equations is absent in all the eight books. Every book has presented *four different solving methods*: null-factor law (factorization) and square root method for solving simple quadratic equations; completing the square method and the quadratic formula (also called *the pq-formula* in Swedish mathematics classrooms) for completing quadratic equations of the *type of* $x^2 + px + q = 0$ $(p \neq 0, q \neq 0)$. *All the textbooks put emphasis on the two methods: completing the square and the quadratic formula* which means that these two content elements are explained in more detail and take more space than other solving method presentations. Factorization is given little space among and is related to the null-factor law. Factorization is used only for solving simple quadratic equations by making use of the distributive law, the square rule, and the difference-of-two squares formula reversely. The factorization method for general quadratic equations presented in Chapter 3.2.2b) is absent in all the eight textbooks.

The findings from the third round of analyses have answered the research question 1b). *Connections between quadratic equations and quadratic functions* are shown in two ways:

A. Quadratic equations are presented in the chapter of quadratic functions in the context of finding the symmetry line and *x*-intercepts of a quadratic function with an algebraic method through solving quadratic equations. However, it is presented as a mathematical tool and has no other influence on the presentation of quadratic functions.
B. In some textbooks (two of the eight investigated textbooks) quadratic equations are introduced by making connections with quadratic functional graphs.

In addition, quadratic equations and functions are treated separately in two different chapters. Application of quadratic equations and functions as mathematical models are presented in some textbooks. In four of the eight textbooks, quadratic equations are introduced through connections with geometry.

Two kinds of organization orders are found depending on if quadratic functions are presented before or after the presentation of quadratic equations. The third round of analyses shows that more than half of the investigated textbooks (five of the eight textbooks) present quadratic equations before quadratic functions. Three of the eight textbooks present quadratic functions before quadratic equations but only one out of the three explicitly explains the relationship between quadratic functions and equations. *The relationship is that quadratic equations can be solved through finding coordinates of x-intercepts in a graph of a quadratic function* (Szabo, Larson, Viklund, & Marklund, 2008).

The second round of analyses shows that *the algebra content elements are presented in such an order that every new algebra content element builds on the previous one*. For example, multiplication of two binomials is based on the four operational rules for simplifying polynomials; the difference-of-two squares formula and the square rule formula are the main techniques applied for factorization through using distributive law inversely. *The last element in this algebra content is always the presentation of the quadratic formula which can be regarded as a final goal of the progression*. The same result was also found in the first round of analyses.

It was found in the second round of analyses that *factorization in these three textbooks is presented for factoring algebra expressions, either of the type:* $a(b+c)=ab+ac$ *or* a^2-b^2, $a^2 \pm 2ab+b^2$. It was also found that *factorization presented as an operational procedure and lacks a detail definition*. This finding is identical to the result from the first and third rounds analyses derived from the other textbooks. For example, factorization is described as breaking out the greatest common factor (GCF) from a quadratic polynomial in the first textbook (Björk et al., 2000) where a third degree polynomial $9x^3-18x^2+36x$ is given for factoring. The factoring procedure of this polynomial is described in the following five steps: $3(3x^3-6x^2+12x)$; $9(x^3-2x^2+4x)$; $x(9x^2-18x+36)$; $3x(3x^2-6x+12)$; $9x(x^2-2x+4)$. After the demonstration of these steps, the last result of factorization is regarded as the most correct one according to the textbook: "[…] by factorization, we prefer the last case because we break out the greatest common factor (9*x*) as much as possible" (Björk et al., 2000, p. 85).

In a second textbook (Alfredsson et al., 2007), factorization is presented in relation to using the difference-of-two squares formula and square rules in reverse:

> We can write a number or an expression as a product of factors. When we write: $1001 = 7 \cdot 11 \cdot 13$, we factorize the number 1001. $x^2 - 9 = (x+3)(x-3)$ We factorize the polynomial $x^2 - 9$. Factorizing of algebraic expressions can be used for simplifying and solving equations. We show two methods for factorizing polynomials.
> 1. Factoring out the greatest common factor. $2x^4 + 6x^3 - 4x^2 = 2x^2(x^2 + 3x - 2)$.
> 2. Using the difference-of-two squares formula and the square rules inversely. (Alfredsson et al., 2007, p. 21)

$$\boxed{\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ a^2 + 2ab + b^2 &= (a+b)^2 \\ a^2 - 2ab + b^2 &= (a-b)^2 \end{aligned}}$$

Figure 12. The difference-of-two squares formula and the square rules written in reverse in *Matematik 4000 B* (Alfredsson et al., 2007, p. 21).

In the third textbook (Gennow et al., 2005b), factorization is described as breaking out the greatest common factor through using distributive law in reverse:

> When a factor is multiplied in a pair of parentheses, the distributive law is used, that is $a(b + c) = ab + ac$ The terms on the right side of the equal sign have a common factor a. That means the distributive law can be used backwards (in reverse) if there are common factors in $ab + ac = a(b + c)$. The expression is divided into factors or factorized. It is even said that the factor a has been "broken out" (Gennow et al., 2005b, p. 88).

To compare these three different descriptions of factorization with an established definition, I here use a quotation from the website Wikipedia:

> Factorization (also factorisation in British English) or factoring is the decomposition of an object (for example, a number, a polynomial, or a matrix) into a product of other objects, or factors, which when multiplied together give the original. For example, the number 15 factors into primes as 3 × 5, and the polynomial $x^2 - 4$ factors as $(x-2)(x+2)$. In all cases, a product of simpler objects is obtained.
>
> The aim of factoring is usually to reduce something to "basic building blocks," such as numbers to prime numbers, or polynomials to irreducible polynomials. Factoring integers is covered by the fundamental theorem of arithmetic and factoring polynomials by the fundamental theorem of algebra. Viète's formulas relate the coefficients of a polynomial to its roots.
>
> The opposite of factorization is expansion. There is the process of multiplying together factors to recreate the original, "expanded" polynomial.
>
> Integer factorization for large integers appears to be a difficult problem. There is no known method to carry it out quickly… (Wikipedia, 2010).

A detailed definition of factorization is absent in all the investigated textbooks.
The common result obtained from the first three rounds of analyses shows that *factorization in these textbooks is applied only for solving simple quadratic equations of the type* $ax^2 + bx = 0$ ($a \neq 0$; $b \neq 0$), or an expanded form derived from the difference-of-two squares formula such as $x^2 - 1 = 0$, or an expanded form of square rules such as $x^2 \pm 2x + 1 = 0$. There is an absence of factoring general quadratic equations with small integers, for example $2x^2 + x - 6 = (2x-3)(x-2)$ though it is possible to use this absent factorization in some exercises. The second round of analyses has found that 69% of the quadratic equations in an exercise set in *Matematik 3000 B* (Björk et al., 2000) and 63% of the quadratic equations in another exercise set in *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007) could have been solved by using the absent factorization method. Using the absent factorization is more effective than using the quadratic formula to solve the quadratic equations because the procedural steps in the factorization method are fewer than in the quadratic formula.

6.2 What aspects of pedagogical content knowledge can be traced in a Swedish upper secondary school textbook?

In order to present the results derived from the fourth round of analyses on textbook *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007), I here first introduce the structure of the investigated textbook in 6.2.1 before I present the results in 6.2.2, 6.2.3 and 6.2.4 according to the order of the three sub-questions of the second research question. A short conclusion will be given after every result part.

6.2.1 The structure of *Matematik 4000 B* (the Blue book)

Matematik 4000 B (the Blue book) (Alfredsson et al., 2007) is used for the science and technology programs in the Mathematics B course at Swedish upper secondary school. As earlier mentioned, there is also a teacher's material or teacher's handbook for this book although the publisher says that there is not. However, there is a teacher's handbook for *Matematik 3000 B* (Björk et al., 2000) and a teacher's handbook for *Matematik 4000 A* (Alfredsson, Erixon, & Heikne, 2008a). I use these two teacher's handbooks as references when I am analyzing *Matematik 4000 B* (the Blue book) which lacks a teacher's handbook.
Matematik 4000 B (the Blue book) (Alfredsson et al., 2007) has three chapters in total. Every chapter consists of: an introduction activity; mathematical units where every unit contains presentations of mathematical contents, solved mathematical problems as illustrated examples, three-level exercises; activities (four kinds: investigating, discovering, doing laboratory work and discussing) as well as historical notes at the end of each chapter; homework exercises including previous mathematical contents; a summary of the whole chapter; mixed three-level tasks consisting of two tests (also including contents from the previous chapter) which are similar to national tests; *Problems for everybody* for students independent work; and finally repetition exercises at the end of the book. The following diagram illustrates the structure:

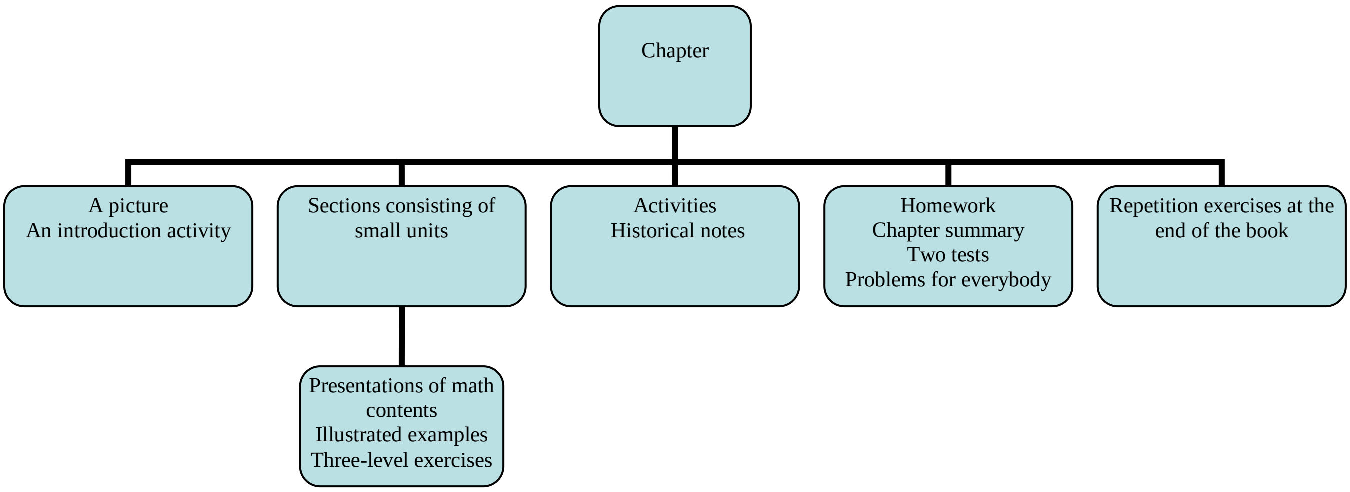


Figure 11: A brief structure of the content in every chapter in *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007).

It is said in the teacher's handbook for *Matematik 4000 A*, that the different mathematical moments in *Matematik 4000 A* are divided into small units to function as lesson plans for the teachers and therefore is more effective for teaching (Alfredsson et al., 2008a). These small units can be interpreted as presentations of mathematical content (textual presentation and examples) with relevant exercises in every section of *Matematik 4000 B*. This analysis follows the division order according to the textbook, namely chapters → sections → units. The whole textbook contains three chapters: 1. Algebra and geometry; 2. Functions; 3. Probabilities and statistics.

Since this analysis includes only algebraic content, the analysis focus will be put on the major part of the first chapter and a small part of the second chapter in the textbook.

Every chapter begins with a picture and an introduction activity, consists of sections of small units as the core content, then ends with a discussion activity, homework exercises, a summary of the whole chapter, two mixed exercises (A + B) and *problems for everybody*. In other words, between the introduction activity and the general exercises at the end of the chapter, the sections are filled up with two kinds of contents: one is in the form of activities like a discovering activity or an investigating activity; and one is the core content which consists of a textual presentation of mathematical content with examples and relevant exercises.

The chapter on algebra, excluding geometry, in the textbook consists of ten units and three activities as well as a presentation of some important mathematicians in algebra history. Every unit presents an algebra topic covering certain algebra content. The algebra content in the ten units is:

1. Introduction of different polynomials and terms related to polynomials
2. The value of a polynomial (decided by the value of the variable in the polynomial)
3. Polynomial computing laws: commutative law; associative law; distributive law; parenthesis rule

4. Multiplication of two binomials
5. The difference-of-two squares formula and square rules
6. Factorization by using the difference-of-two squares formula and square rules inversely
7. Using the square root method and null-factor law to solve simple quadratic equations
8. Using the approach of completing the square to solve pq-quadratic equations of the type $x^2 + px + q = 0, p \neq 0, q \neq 0$.
9. The general formula called quadratic formula or the pq-formula used for solving quadratic equations of the types $x^2 + px + q = 0, p \neq 0, q \neq 0$ and $ax^2 + bx + c = 0, a \neq 0$.
10. Application of Pythagoras' theorem and quadratic equations in solving geometrical and "real world" problems

At the beginning of the chapter, there is an introduction activity (investigating activity) containing four exercises related to geometrical representations and algebra rules.

Between units 6 and 7, a group-work activity (investigating activity) is presented concerning finding and generalizing as well as applying and proving rules or algebraic models for calculating multiplications of big numbers.

At the end of unit 10, another group-work activity (discovering activity) is presented concerning finding and proving relations between roots, coefficients and constants of quadratic equations; generalizing the rules from the findings; and applying the rules to solve quadratic equations.

Algebraic history includes five mathematicians and their solving methods for solving the third-, fourth- and fifth-degree equations presented after the discovering activity.

There are in total nine units with mathematical content textual presentations including exercises and one unit on algebraic applications as well as three activities in the whole presentation related to solving quadratic equations as a final goal. Among these 10 units, 151 exercises containing A, B, and C levels and three activities are offered for students to practice the newly presented mathematical content. At the end of the first chapter, more exercises are provided. They are: one discussion activity, one set of homework exercises, two mixed tests and an exercise set of *Problems for everybody* which is related to both algebra and geometry.

6.2.2 How is mathematical content presented or explained?

The history of mathematical ideas is an important aspect of mathematics that should permeate mathematics teaching, according to the Swedish national mathematics curriculum at upper secondary schools (Skolverket, 2010). In *Matematik 4000 B*, the algebra development in history is reflected in some presentations.

This study has found that some algebra content presented in the textbook (Alfredsson et al., 2007) are illustrated and explained by *relating to algebra history* since they can be traced back to the historical idea of solving quadratic equations by *"cut and paste" geometry* (Derbyshire, 2006, pp. 25-27). In the textbook, algebra content like the distributive law and expanding the product of two binomials as well as completing the square method are illustrated by the use of cut and paste geometry. The application of this historical idea as a pedagogical approach is embedded in the beginning of the chapter on algebra and appears several times in the chapter. The details will be presented in section 6.2.4 on embedded teaching trajectory.

Historically related pedagogy is applied in the background presentation following the part on quadratic equations in the textbook. It is done through a presentation of five mathematicians and the third, fourth and fifth degree-equations solved by them. For example, the origin of solving quadratic equations is mentioned in one sentence: "Babylonian clay tablets have shown that solving quadratic equations was known 4000 years ago" (Alfredsson, 2008 p. 32). After the presentations of different mathematicians, three algebraic word problems are offered with the intention to inspire the students to solve those problems with the third degree solving formula. An embedded PCK is exposed that is intended to encourage students to dare use formulas as solving tools, even if the formulas are even beyond their current comprehension.

Historically related pedagogy is also reflected in some word problems such as an old Chinese classical problem of using Pythagoras theorem to solve a practical problem and a Babylonian clay tablet problem.

Besides completing the square method, other *algebra content, including the quadratic formula and factorization as well as a mathematical activity in the textbook, have their historical roots in different algebra development stages as* presented in Chapter 3 in this thesis. The completing the square method originates from the cut-and-paste geometrical ideas written on clay tablets in ancient Babylon. The same method was also used by al-Khwarizmi at the syncopated stage (Kvasz, 2006). The quadratic formula also has its origin in ancient Babylon but was developed by Euclid. Both solving methods belong to the rhetorical stage in algebra history (Kvatz, 2007). The order is that geometrical ideas generally came before the symbolic representations. In the textbook, a mathematical activity on discovering the relationship between roots and coefficients of a quadratic equation has its historical connection with the French mathematician François Viète's finding of $(x-\alpha)(x-\beta)=0$ in the quadratic equation $x^2+px+q=0$ if $x=\alpha; x=\beta$ (Derbyshire, 2006). It was Viète who used algebraic symbols to write the quadratic formula (Olteanu, 2007). The symbolic quadratic formula and the relationship between roots and coefficients of an equation belong to the symbolic stage (Kvatz and Barton, 2007). The symbolic stage came after the rhetorical and syncopated stage. Factorization is not directly mentioned by Kvatz and Barton (2007) though it belongs to the purely abstract stage when it is related to polynomials in the field of algebra structure (Durbin, 1992). Thus, the algebra content concerning completing the square method, the quadratic formula, and the relationship between roots and coefficients of a quadratic equation, has its roots in historical algebra development stages.

It is found that *a mathematical concept in the textbook is often presented in the context of a mathematical example* to illustrate the meaning of the concept. For example, the concept of the fourth degree polynomial (Alfredsson, et al., 2007, p. 8) is illustrated in an example (see Figure 13).

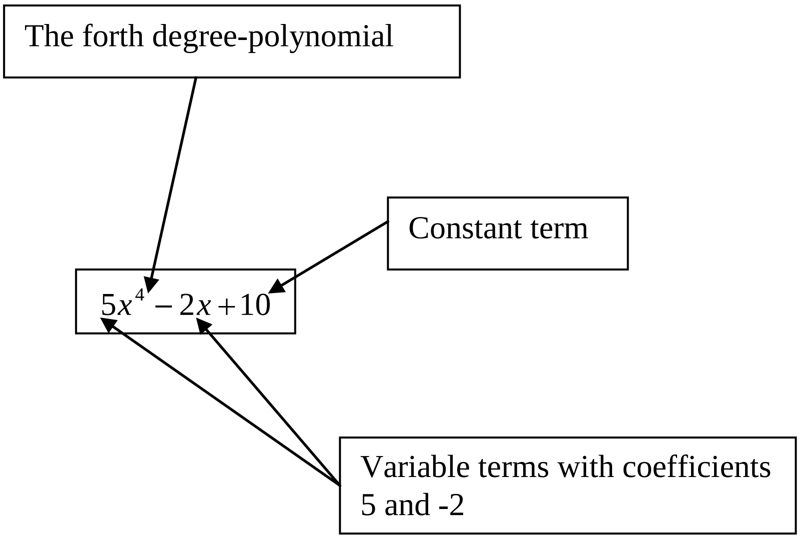


Figure 13. An example of a fourth degree polynomial on page 8 in *Matematik 4000 B.*

Before this example, the short definitions of polynomials, terms, coefficients and polynomial's degree are presented by three examples to prepare a background for the presentation of a fourth degree polynomial. After this example, the other two examples of non-polynomials are given to show the contrast between polynomials and non-polynomials.

This pedagogical approach of presenting the new algebra content in the context of mathematical examples has been explored in this analysis. *A short and everyday language* is often used to explain a new concept or rule, for example, when the multiplication of two binomials is introduced, a rule of signs is given as follows: "the same signs give plus, the different signs give minus" (Alfredsson, et al., 2007, p. 15).

The results indicate that some mathematical concepts and expressions need be improved to avoid confusion. They involve in the following points:

- *Explicit and detailed definitions are needed.* For example, the expressions of polynomials and polynomial functions as well as polynomial terms appear on the same page when polynomial as a concept is introduced. No distinction among them is given. They are all presented in the context of examples with short descriptions as in: "A polynomial is a sum of terms"; "Every term is either a variable term or a constant term"; "A variable term of a polynomial is a product of a number which is called coefficient, and the variable with a positive whole number as the exponent" etc (Alfredsson et al., 2007, p. 8). Factorization and quadratic equations are presented without definitions but are illustrated in the examples instead. Some mathematical concepts suddenly appear in the exercises but have never been presented in the previous theoretical parts. Examples here are the terms of double roots and discriminant of quadratic equations.

- *Some mathematical expressions need to be more precisely explained*. As mentioned above, there is a mixed use of polynomials and polynomial functions as well as quadratic equations and quadratic functions in the textbook. Another confusing presentation concerns the operational sign and number sign before an integer when the the pq-formula is explained in words in Unit 9. Mixing the categories of two signs can cause conceptual confusion when applying the the pq-formula to solving quadratic equations. The verbal explanations for the procedures embodied in the the pq-formula $x = -(p/2) \pm \sqrt{[(p/2)^2 - q)]}$ are described as: "*x* = (*half of the coefficient to x with inverted sign*) ± √ [(*square half of* *coefficient to x*) − (*the constant term with inverted sign*)]" (Alfredsson et al., 2007, p. 28). The last comment on the procedure of minus *q* in the the pq-formula is explained as the constant term with inverted sign. This phrase is written after the subtraction sign which can be interpreted as if the constant term with switched sign excludes the subtraction sign. Actually, the subtraction sign should be explicitly explained if it is an operational sign or number sign, otherwise the description of the explanation is incomprehensible according to the book.

There is a problem for the expression of a sign (either minus or plus) before a constant. The sign before the constant in the *pq*-quadratic equation $x^2 + px + q = 0$ is an operational sign representing addition or subtraction while the sign before symbol *q* in the description of the explanation mentioned above means the sign of a constant representing the negative or positive real number. The word of "sign" here has a double meaning. Which meaning does the sign in the textbook refer to? Representing operations or number domains? Example 1220 has the quadratic equation: $x^2 + 6x - 16 = 0$. To solve this equation, the first step is carried out by the book according to the quadratic formula: x = - (6/2) ± √ [(6/2)² + 16)]. In the illustration of the procedure, addition of 16 in the square root is explained as "the constant term with inverted sign" (Alfredsson et al., 2007, p. 29). The sign before 16 is regarded as a positive sign as opposed to the subtraction sign before 16 in the equation. If the sign represents only operations, the meaning of the sign in the solving form is changed to represent a positive number. Where is the operational sign in this case? The dual meanings of the sign could be confusing.

- *The passage between the two sections needs to be explained.* After the first section including the first six units on the polynomial concept and operational rules, the second section (unit 7, 8, 9 and 10) starts with solving simple quadratic equations. The contexts have changed from polynomials to quadratic equations, but there are no explanations as to why quadratic equations are presented in this section and how this section relates to the first section.

Conclusion

Historically related pedagogy as an aspect of the PCK is built into the presentation of algebra content including geometrical models, the quadratic formula and factorization as well as a mathematical activity and real world problems. The mathematical concepts in the chapter of algebra are often presented in the context of an example. The short and everyday language is used for explaining the mathematical concepts, however some terms and formulas need explicit explanations.

6.2.3 What is the character and function of the presented examples and exercises?

Comparing algebra content textual presentations with activities and exercises in the textbook, the analysis shows that the latter takes up a big percentage of the total mathematical content.

Among the 151 exercises divided into A, B and C levels, *more than half of exercises are intended for routine practice* which means that students are expected to practice what has just been presented in the previous units. The exercises can often be carried out without much difficulty by following operational rules or methods presented in the related content presentations in the units precisely. These exercises have "*lower goals* – rote skills, simple rules and algorithms, definitions" (De Lange, 1996, p. 89). They emphasize procedures and are instructed with simple and short expressions. For example, "Simplify," "Multiply in," "Solve the equations," "Expand" etc. Short instructions are of the kind "Explain how you simplify…," "What does XX represent?" and so on.

Sometimes however, some exercises are not necessarily simple, and those are exercises that demand a more conceptual understanding. For example, Exercise 1150 in Unit 5 on the difference-of-two squares formula and quadratic rules: "Which rule can be illustrated by the following figure? Explain" (Alfredsson et al., 2007, p. 19).

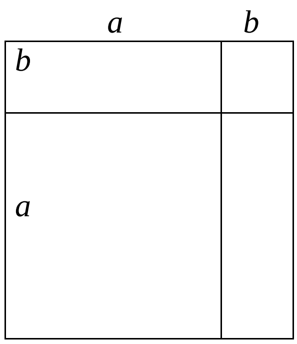


Figure 14. Exercise 1150 on page 19 in *Matematik 4000 B.*

Although this exercise is on the A level (lower-goal level), mathematical reasoning and explanations are required. The lower-goal exercises do not mean that easier exercises are less important since more than half of the exercises are of such a kind. The pedagogical intention is to reach an education goal where all students can be good at all levels (De Lange, 1996).

The other half (or less than half) of the exercises can be characterized as having the following aspects of embedded pedagogical content knowledge:

A. Exercises are provided to develop students' *structure sense of polynomials or quadratic equations*. They often require students to find unknown coefficients or constants by given values of polynomials or quadratic equations. Exercise 1111 serves as an example of this: "Let $p(x) = ax^2 - 2ax + 11a$. Determine *a* if *p*(-2) = 5" (Alfredsson et al., 2007, p. 10). In this exercise, *a* is a coefficient to variable *x*. The value of the polynomial and variable are given. In order to find *a*, the polynomial is rewritten into an equation with the unknown *a*. By solving the equation, the value of the coefficient *a* can be obtained. The essential step in order to carry out this exercise is to recognize the polynomial structure and change the unknown from a variable *x* to unknown *a*.

   Some of the exercises of this kind require that students analyze and judge the situation of the roots of a given quadratic equation. Exercise 1216: "For which value does the equation $x^2 - 14x + a = 0$ lack real solutions?" (Alfredsson et al., 2007, p. 27). The aim of such exercises is to find relationship between roots, coefficients and constants. Mathematical analysis, reasoning and judgment are demanded for some of this kind of exercises. Exercise 1228:

   > Louis and Nille want to solve the equation $x^2 + x - 2 = 0$ with the quadratic formula.
   > Louis:"One of the coefficients of x is missing, p is 0."

Nille: “We have one x, p is 1.”
Who is right? Solve the equation. (Alfredsson et al., 2007, p. 30)

This exercise tests if students understand $x = 1x = 1 \cdot x$. A misconception of $x = 0x$ is possible. The pedagogical purpose of this exercise is to check students’ concept of coefficient with the value of one, which is implicit and never written before an unknown or variable.

B. Some exercises are intended for practicing *reversed operation procedures*, which means setting up an equation to fit the given roots. This kind of exercise often overlaps the category A about practicing algebra structure sense. An example is exercise 1206: “Give your own example of a quadratic equation with the following solutions: a) $x = 0$ and $x = 12$ b) $x = 4$ and $x = 5$” (Alfredsson et al., 2007, p. 25). This exercise requires students to be able to use the null-factor law in reverse to find the quadratic equations that are based on $(x-0)(x-12)=0$ and $(x - 4)(x - 5) = 0$ respectively. The complexity of this kind of exercises varies. Some are simple operations such as factorization exercises while the construction of some is more complex. For example, Exercise 1236:

> Indra and Fanny are going to solve an equation of the kind $x^2+bx+c=0$. Indra writes the second term (bx) wrong and gets the solutions -6 and 1. Fanny writes the last term (c) wrong and gets the solutions 2 and 3. Which equation are they trying to solve? (Alfredsson et al., 2007, p. 30)

This exercise examines algebra structure sense by rewriting the new equations based on given roots. The process is complicated and includes five steps: setting up quadratic equations; setting up two linear equation systems; solving equation systems and finding the value of b and c; setting up new equations according to the obtained value of b and c; comparing the new equations and finding the correct one.

C. Some exercises are *related to conceptual understanding of newly presented algebra content*. Those exercises aim at examining students’ understanding of the newly learned operational rules, methods or algebraic concepts but they form a minority group. One example here is exercise 1227 (Alfredsson et al., 2007, p. 30):

> Give an example of a quadratic equation which can be solved by the a) the square root method b) the null-factor law (factoring) c) the quadratic formula.

Students need to be familiar with which type of quadratic equations that adopts which solving method. This requires that the students are able to generalize different types of quadratic equations:
a) $(x+m)^2=n$ b) $ax^2+bx=0$ c) $ax^2+bx+c=0$ d) $x^2+px+q=0$.
Conceptual understanding of algebraic representations of different types of quadratic equations and knowledge of different solving methods as well as operational rules are demanded.

D. Some exercises are *word problems related to a real world context* or other subjects such as physics or chemistry, or numerical computations, geometrical problems or even historical problems. They are mainly placed in the first unit of polynomials and the last unit of algebra applications. Ten out of 151 exercises are real world context problems and make up a minority group. The aim of such problems is to provide the students with opportunities of first interpreting the problems, then setting up quadratic polynomials or quadratic equations and finally solving them. Another aim is to encourage the use of algebraic quadratic expressions or equations as tools to solve mathematical problems. Among these word problems, there are exercises that are short and simple in their construction but also complex ones. Exercise 1140 is one example: “Is there any positive value for *a* which gives a rectangle with the sides ($x + a$) and ($x - a$) the same area as the area of a square with every side *x*?” (Alfredsson et al., p. 17).

This exercise has a simple construction and requires the setting up of a quadratic equation and then analysis of the structure of the equation. However, its aim is also to examine the students’ structure sense. The word problems related to real world contexts constructed in the most complicated way are a couple of historical problems from ancient China and Bagdad. Exercise1251 (Alfredsson et al., 2007, p. 36):

> In a 2000-year-old Chinese writing *Nine chapters calculation art* (Nine chapters arithmetic), we find the following problem: “In the middle of a square lake with its side at *s* meters, there is a reed growing *h* meters over the surface of the lake. If the reed is pulled toward the side of the lake, it exactly reaches the surface of the lake. The depth of the lake is *d* meters.” Show that
>
> $$d = \frac{s^2}{8h} - \frac{h}{2}$$

Although it is a mathematical geometrical proof, this word problem requires five steps to finish the proof: reading the word problem and interpreting it; drawing a picture of the lake and the reed marked with given data; drawing a right-angled triangle derived from the interpretations of the first two steps; setting up a quadratic equation with the help of the third step and Pythagoras’ theorem; algorithmically calculating the quadratic equations by only using symbols and then getting the proof done. Interpreting the words into the drawing is an essential but difficult step requiring students’ geometrical imaginations. Such word problems are challenging since none of the previous examples are of this kind.

E. A few exercises are about *mathematical proofs*, which often contain operations of abstract symbols related to algebraic formula or operational rules. This can be seen in exercise 1234 (Alfredsson et al., 2007, p. 30):

> In other countries, the “abc-formula” is used instead of our “the pq-formula”. Show that the equation $ax^2 + bx + c = 0$ has the solutions
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

F. A couple of the exercises or activities provided in these ten units *function as completion of the absent mathematical content from the textual presentation parts* so that the related mathematical content can be developed to an advanced level by students doing the exercises. Exercise 1155 (Alfredsson et al., 2007, p. 19) serves as an example here:

> a) One side of a square is *x* cm. If the length of two sides increases with 5 cm, how much does the square’s area increase?

b) In another square, one side of the square increases the length with 5 cm and the other side decreases in length with 5 cm. How does this change the area?

The second sub-exercise of this exercise requires students to actually analyze and experience how the difference-of-two squares formula works in a varied square model, something which is not presented in the previous content presentation parts. So the second sub-exercise plays a completing content role here. Another example is a discovering activity on finding relationships between roots and coefficients, which have not been presented in the previous content presentations. At the same time, this activity indirectly presents the factorization method to be used for solving pq-quadratic equations. Such factorization is absent in the textbook. Therefore, this discovering activity functions as completing the absent mathematics content. Exercise 1234 above also has the same function: to develop the the pq-formula into an abc-formula for solving the quadratic equation: $ax^2 + bx + c = 0$.

G. A couple of exercises have a function of *linking the content* of the previous unit to the content of the new unit in the embedded sequences. Exercise 1219, for example, reads: "Use completing the square and determine the solutions to the equation $x^2 + px + q = 0$" (Alfredsson et al., 2007, p. 27). This exercise introduces the content of the next unit on *the pq-formula* and provides students with an opportunity to work out this formula themselves first before the next unit. At the same time this activity is based on the content of completing the square method in the previous unit. So it functions as a connection activity between the two units next to each other.

Among these exercises, the most different and advanced exercises are *real world problems* even if the number of such exercises is relatively low in this chapter (16 real world problems in total, 10 from the 10 units and 6 from the rest of the chapter). The real world problems are not genuine applications like mathematics used for technical sciences or economy etc (De Lange, 1996), but they are all formulated by linking to contexts which learners may feel familiar with or can imagine. This includes for example changes in shopping prices, change of the area of a lawn, and the size of a television screen and so on. This type of context is related to daily life. For example, "Malin has two jugs which can hold 3 and 5 liters. How can she measure exactly 4 liters of water with these jugs?" (Alfredsson et al., 2007, p. 67). This problem can be solved without relating to algebra content presented in this book and it seems intended to motivate students' mathematical creativity.

Other subjects, such as physics and biology, have also been used to formulate mathematics problems, which have been presented as examples in the presentation of polynomials. One example: "A ball is thrown away upwards at the speed of 30 m/s. What is the speed after *x* seconds?" (Alfredsson et al., 2007, p. 8). Such problems are often provided with given mathematical expressions and require students to interpret or compute the speed. The pedagogical purpose is to give a message to students that mathematics can be used in other subjects.

Real world problems become difficult when they do not have given algebra expressions and require students to understand the problem so that they can first interpret it into an algebra expression before they compute the result.

One example is the ancient Chinese problem mentioned above, requiring five steps to establish the mathematical proof. Mathematics application is reflected in this problem because of the use of Pythagoras' theorem and setting up a quadratic equation by finding the equivalent relationship in the solving process. A similar solving process has not been presented in previous examples. Although the real world problems are few, there is an embedded pedagogical intention to foster students' mathematics competence of translating word problems to algebraic expressions, which is to apply mathematics.

The result shows that *mathematics application in two types of contexts* has been explored in the exercises. These two types of contexts are *real world and pure mathematics*. The word problems related to real world contexts have been mentioned above as real world problems. Another type refers to two mathematics activities presented in the textbook: one investigating activity and one discovering activity. The explanation of the two activities related to application of mathematics can be found below:

The investigating activity includes four exercises. The pedagogical intention is to make students to find an operational rule for numerical multiplications through generalizing the patterns of the numerical products. The first exercise is (Alfredsson et al., 2007, p. 23):

1. Study the square of every number ending by 5

   $5^2 = 15 \cdot 15 = 225$

   $25^2 = 25 \cdot 25 = 625$

   $35^2 = 35 \cdot 35 = 1225$

   $45^2 = 45 \cdot 45 = 2025$

a) What do all these results have in common?
b) Can you find a pattern? Formulate an easy rule for the squares of the numbers ended by 5.
c) What about 75^2 according to the rule? Check with a calculator.
d) Can you explain why the rule works?

Students are first required to find the common character of the results from the four numerical products; then formulate an algebra rule or expression based on the pattern; later test the rule with another example and control the result with a calculator and finally explain why the rule works in these cases. According to the given patterns, all the expressions can be generalized as a perfect square: $(x+5)^2$. The second task is similar to the first one. The third exercise requires students to find the rule for the given products: $12 \cdot 18 = 216$, $23 \cdot 27 = 621$ and explain why the rule works. The instruction of the third exercise says that the product of two numbers has the same ten-digit numbers and the sum of their single-digit numbers is 10. To find the rule for this exercise, students need to be creative since the three learned rules from the previous part are not useful in this exercise.

My analysis is thus: the products can be written as $12 \cdot 18 = (10 + 2)(10 + 8)$; and $23 \cdot 27 = (20 + 3)(20 + 7)$. It is said that the sums of the two single-digits are 10: $2 + 8 = 10$ and $3 + 7 = 10$ respectively. The products of these two single digits are $2 \cdot 8 = 16$ and $3 \cdot 7 = 21$ respectively. There is a special relationship between 2, 8, 10, 16 for the product of 12 and 18. The first term in the parentheses are 10, 20, 30 and so on which can be represented by the variable x. When x represents 10, the product can be written as: $12 \cdot 18 = (x + 2)(x + 8) = x^2 + 2x + 8x + 16 = x^2 + 10x + 16$. The other product is $23 \cdot 27 = (20 + 3)(20 + 7) = x^2 + 10x + 21$ when x is 20. The last exercise is about using the square rule for the difference of two numbers.

The three exercises in this investigating activity has *some characters of mathematization* (De Lange, 1996) since the process of carrying out every exercises includes finding something in common according to the given patterns, then formulating a mathematical expression or formula, thereafter applying the formulation. But differing from real world contexts, mathematics application in this activity is carried out in the context of pure mathematics.

The discovering activity requires the students to work in groups. The whole activity consists of seven exercises. The pedagogical purpose of this activity is to make the students to work out the relationship between the coefficients p and q and the roots of the quadratic equation $x^2 + px + q = 0$. The first exercise of the activity requires the students to fill in a table with the two roots to each equation (see Table 3).

Table 3

Coefficients and roots to every quadratic equation in a discovering activity (Alfredsson et al., 2007, p. 30)

Equations	p	q	x_1	x_2
$x^2 - 4x + 3 = 0$	-4	3	3	1
$x^2 - 8x + 15 = 0$	-8	15	3	5
$x^2 + 6x + 8 = 0$	6	8	-2	-4
$x^2 + 12x - 28 = 0$	12	-28	-14	2

Next step, the second exercise requires students to investigate the table and try to find a relationship between the coefficients and the roots, and then formulate it by words and formula. The formula representing the relationship is later applied to more quadratic equations in the next four exercises. The last exercise is about proving the formula.

The relationship between the coefficients and roots can be formulated as: $q = x_1 \cdot x_2$; $p = -(x_1 + x_2)$ which has the historical root in Viète's finding (Derbyshire, 2006). All the quadratic equations in this activity are factorable. They belong to the equation type $x^2 + px + q = 0$ and can be solved by factorization through the null-factor law. But this kind of factorization is absent from the textbook.

The process of the activity includes discovering the relationship between the coefficients and the roots; representing the relationship in a formula; using the formula and proving the formula. *Such procedures share some characters of mathematization but are carried out in the context of pure mathematics, which can be regarded as mathematics application*.

Conclusion

All the exercises are divided into three different levels from easy to difficult. Compared to the mathematical textual presentations, the mathematical exercises and activities as well as problems take up a big space in the chapter of algebra in this book. More than half of the provided exercises have the character of training students' basic mathematical procedures. These are the routine exercises. Such exercises reflect learner-centered design (LCD) pedagogy (Selander, 2003) whose purpose is to adapt various needs from students and encouraging learning. The rest of the exercises and activities as well as problems are provided for fostering students' cognitive complexity (Schmidt et al., 1997). Among these exercises, the PCK aspects are explored by the following cognitive complexity: training algebra structure sense, reversed operational procedure, conceptual understanding of the algebra rules, introducing the advanced activities like mathematics proof and mathematics application (Freudenthal 1991; Van Den Heuvel-Panhuizen, 2003) in both real world problems and pure mathematics contexts. The mathematics activities in pure mathematical contexts have the characters of mathematization (De Lange, 1996). The large amounts of exercises reflect the pedagogy of acquisition of knowledge with activities (Van Dormolen, 1986).

6.2.4 What embedded teaching trajectories are built into the presentations of quadratic equations? How are those trajectories constructed?

The fourth round of the analyses shows that the algebraic content in the textbook of *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007) is presented in a particular order which has its origin in the introduction activity in the beginning of the textbook and the chapter of algebra. The use of the geometrical models, which are often called algebra tiles[17] (Leong et al., 2010; Howden, 1985; Norton, 2007), constructs a few algebra content trajectories (Ferrini-Mundy et al., 2003) as embedded teaching trajectories according to a part-whole relationship. This part–whole relationship helps to develop a progression on learning the distributive law and completing the square rule for solving quadratic equations. By organizing and changing the geometrical models in different ways based on the introduction activity, the textbook has developed a cumulative sequence consisting of the different algebra representations and operational rules with the final goal of presenting solving methods for quadratic equations. In such a way, these teaching trajectories function like sub-trajectories together construct an overall teaching trajectory with the final goal of teaching how to solve quadratic equations by the quadratic formula.

The concept of models in this analysis does not have the same character as Realistic Mathematics Education models (Van Den Heuvel-Panhuizen, 2003) since there is neither any connection with realistic mathematics nor a relation with students directly. However, they have a common function of offering readers (including both teachers and students) visual illustrations in order to make sense for learning the distributive law and completing the square method as well as quadratic expressions, and finally solving quadratic equations. At the same time, the changes of the geometrical models provide a process to develop mathematical concepts from the basic level to a more advanced level. The geometrical visual models have been used and arranged by authors of the textbook apparently with a pedagogical purpose.

How do these geometrical models in the introduction activity in the beginning of the chapter on algebra construct the embedded teaching sub-trajectories and the overall teaching trajectory?

Among the ten units of contextual presentations in the chapter on algebra, the first six units are restricted to presentations of the second degree polynomials while unit 7, 8, and 9 are presentations of quadratic equations. The last unit (10) is about application of mathematics related to quadratic expressions and equations in real world context.

[17] Algebra Tiles are a rectangle or square consisting of a square with a number of equal rectangles and squares. They are used to visually illustrate the procedures of distributive law and factorization.

The overall teaching trajectory starts with the introduction activity in the beginning of the chapter. Evidence from the analyzed data shows that the algebraic content in four exercises in the introduction activity exactly reflects the mathematical topics presented later in the textbook. Furthermore the way in which the four exercises are sequenced, creates the five embedded teaching trajectories which later together construct the overall teaching trajectory.

The function of the introduction activity at the beginning of the chapter plays an important role in constructing the ten units of the algebra chapter according to a cumulative sequence order and creating the five embedded teaching trajectories and the overall teaching trajectory as described below:

Unit 1 and 2

The mathematics content in the first exercise of the introduction activity is about using algebra symbols that represent the lengths of the sides of rectangles to express the area of rectangles. The aim of the exercise is to find an algebraic expression for the area of a rectangle and finding the values of the variable in the expression. Two different sized rectangles in Exercise 1 of the introduction activity (see Appendix 3) illustrate the algebraic representations of a binomial and a monomial based on the area formula of a rectangle. The area formula of a rectangle represents a specific *part and whole relationship* (written as part-whole relationship): length · width = area of a rectangle. The length and width of the two different sides of the rectangle are the parts of the rectangle and represent two linear factors, while the area is regarded as a whole which represents the result of multiplication of the two linear factors.

The content in the first introductory exercise can be seen as a preview of or an introduction to the topics of unit 1 and 2 which are about the basic terms of polynomials and the value of a polynomial. This shapes *the first embedded teaching trajectory* of the basic knowledge of polynomials. The geometrical model of a rectangle area is the most basic model or representation (Figure 15) to start the teaching process with multiplication of two linear factors in forms of algebraic expressions in this overall teaching trajectory of teaching solving quadratic equations.

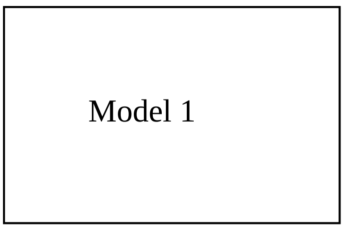


Figure 15. The basic geometrical model of a rectangle area (Model 1) based on the introduction activity.

Algebra expressions and geometrical models are the two forms of representations used in every exercise in the introduction activity and all content presentations in every unit in the chapter of algebra.

Unit 3

When the same rectangle is reorganized, the represented algebraic expression is also changed. Exercise 2 in the introduction activity based on the same rectangle, makes a slight change by dividing it into a square and a small rectangle in order to show the procedure for the distributive law for multiplication of a monomial and a binomial (see Appendix 3). By using the same rectangle in two forms (Model 1 in Figure 15 and Model 2 in Figure 16), an equivalence relationship is set up and the distributive law is illustrated in Exercise 2 by the algebraic expression: $a(a + 2) = a^2 + 2a$ which reflects a part-whole relationship both on the geometrical model (Model 2 in Figure 16) and algebraic representation. Two factors represent the two sides a and $a + 2$ of the rectangle as parts. When they multiply with each other, the result becomes a second degree polynomial, illustrated by the area of the whole rectangle consisting of the two small areas: a square a^2 and a small rectangle $2a$. Again the parts in this case refer to not only the length of the sides but also the two small areas that form the whole or total area.

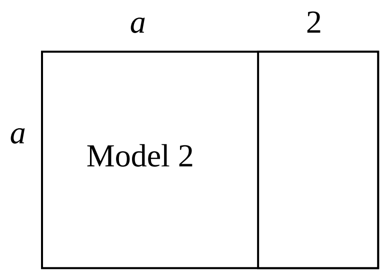


Figure 16. A big rectangle consisting of a small square and a small rectangle constitutes Model 2 in the introduction activity.

The content in Exercise 2 is mapping the content of the distributive law in Unit 3 and Model 2 is used in the same unit. *The second embedded teaching trajectory* is shaped.

Unit 4

With the same part-whole logic, the geometrical model (Model 3 in Figure 17) in Exercise 3 is an extension of Model 2 in Exercise 2. Based on Model 2, Model 3 extends side x with 1 unit so that the whole rectangle consists of a square and three different sized small rectangles (see Appendix 3).

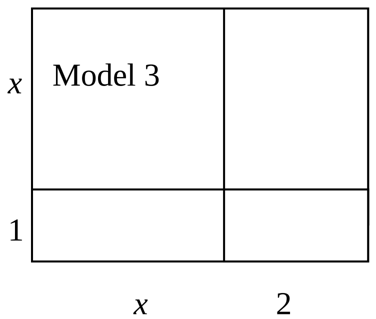


Figure 17. A big rectangle consisting of a small square and three different small rectangles constitutes Model 3 in the introduction activity.

In this extended geometrical model, the algebraic expression represents the multiplication of two binomials with the sides (factors) $x+1$ and $x+2$. Then the whole area is the result of the multiplication of the two factors or sides. It can also be regarded as the sum of the small areas: $(x+1)(x+2)=x^2+1\cdot x+2\cdot x+2\cdot 1=x^2+3x+2$. The content in Exercise 3 suggests the topic of Unit 4: multiplication of two binomials by using the distributive law. The same content here is presented in Unit 4 and *the third embedded teaching trajectory* is shaped. The part-whole relationship continually remains in this exercise.

Unit 5

From Exercise 1 to 3, the developed (or varied) geometrical models are based on one rectangle. But the geometrical model in Exercise 4 is based on a square instead of a rectangle. The change of the models from a rectangle to a square causes a change of the algebra content from two different linear factors' multiplication to two same linear factors' multiplication that is from the distributive law to the square rule. As a result of the change of the model, the square rule is generated in Exercise 4.

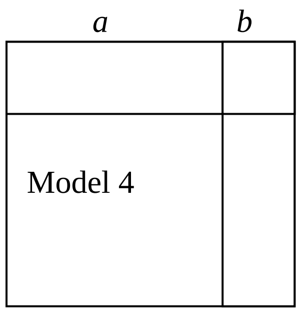


Figure 18. A square consisting of two different small squares and two same rectangles constitutes Model 4 in the introduction activity.

The square rule is generated through the two same sides represented by $(a+b)$ multiplied with itself to generate the whole area of the square. The part-whole relationship continually remains in terms of area and sides. The whole area of the square is made up a two different sized squares and two equal sized small rectangles (see Figure 18). Multiplication of the two same sides can be seen as a multiplication of the two same factors: $(a+b)$ and $(a+b)$ which are the parts; the result of the multiplication is the value of the whole area as a whole. In such a way, the geometrical model in Exercise 4 illustrates the procedure of the square rule and makes the connection between geometrical representation and algebraic representation. The procedure is shown by adding all the small areas: a^2, ab, ab, b^2 into a big area $(a + b)^2$ in accordance with the part-whole relationship. However, this time the equation is ordered in the opposite way, with the whole on the left side and the parts on the right side: $(a + b)^2 = a^2 + 2ab + b^2$. This content in Exercise 4 is mapping the part of the content in Unit 5. Thus *the fourth embedded teaching trajectory* is shaped.

Unit 8

The geometrical Model 2 (Figure 16) in Exercise 2 and Model 4 (Figure 18) in Exercise 4 later appear in Unit 8 when using completing the square method to solve general quadratic equations. These two models together with an added geometrical model (see Figure 19) are applied to illustrate the process of completing the square method, which is the content in

Unit 8. Again, the procedure of completing the bigger square is based on the part-whole relationship. Through removing the two equal rectangles as parts to the two sides of the first square, the second model is derived (in the middle in Figure 19). Then adding a small square in the right corner, the big square is completed which is Model 4. The sub-areas as the parts of the whole big square include the two different sized squares and two equal sized rectangles. *The fifth embedded teaching trajectory* is shaped by linking Model 2 and 4 in the introduction activity with Unit 8.

Therefore this introduction activity has built up the essential algebra content presented in units 1 to 5 and Unit 8. The four exercises and their geometrical models predict how algebra content in this textbook will be organized and therefore provide structures for the embedded teaching trajectories relating to the first five units and Unit 8. The content in Exercise 1 of the introduction activity connects to the topics in Unit 1 and 2 to start the first teaching trajectory with basic concepts of polynomials. The content in Exercise 2 reflects the content focus on the distributive law in Unit 3, which constitutes the second trajectory. The content in Exercise 3 connects to the topic of multiplication of two binomials in Unit 4 and the third trajectory is displayed. Exercise 4 and the half of Unit 5 share the same content on the square rule, which describes the fourth trajectory. The combination of the geometrical Model 2 and 4 relates to the content of Unit 8: completing the square method. The fifth trajectory is explored. The four geometrical models in these four exercises of the introduction activity bridge between the introduction activity and the respective units, displaying a part-whole relationship. In such a way, five trajectories are built for the teaching of polynomials and polynomial computing rules as well as completing the square method. These five embedded teaching trajectories together build up an overall teaching trajectory for teaching quadratic equations since they have prepared the basic knowledge of solving quadratic equations with a general solving formula: the quadratic formula. The varied geometrical models in the four exercises generate three generalized algebraic representations:
$a(a+b)=a^2+ab$ illustrated by Model 2 (Figure 16);
$(a+b)(a+c)=a^2+a(b+c)+bc$ illustrated by Model 3 (Figure 17);
$(a+b)^2=a^2+2ab+b^2$ represented by Model 4 (Figure 18).
These three generalized algebra representations and respective geometrical models represent the distributive law in Unit 3, multiplication of two binomials in Unit 4 and the square rule in Unit 5.

Unit 6

The part-whole relationship remains in these models in the respective units until the presentation of factorization in Unit 6. The part-whole relationship ends with factorization and as a consequence the geometrical models cease being used. Why does the part-whole relationship stop here? My interpretation is that the operational procedure of factorization is to use distributive law and square rules as well as the difference-of- two squares formula in reverse. This means that the algebraic expressions derived from Model 2, 3 and 4 can be factorized by the reversed operations:
$a^2+ab=a(a+b)$;
$a^2+a(b+c)+bc=(a+b)(a+c)$;
$a^2+2ab+b^2=(a+b)(a+b)$.
In this case, the relationship is not from the parts to a whole logic but from the whole to parts. On the other hand, it does not mean that the geometrical models (Model 2, 3 and 4) are not

useful for illustrating factorization. On the contrary, these three models can still be used to illustrate the procedure of factorization in a similar way to the use of algebra tiles (Leong et al., 2010) shown in Figure 10 in Chapter 3.2.

The analyses show that these four geometrical models (1-4) have not only been applied to illustrate the abstract algebra expressions as tools to compute the area of rectangles of different kinds but also to represent and clarify the essential algebraic content – the distributive law, multiplication of two binomials, the square rule, completing the square method – in this book. In such a way, *these geometrical models build up to a process for an overall hypothetical teaching trajectory containing five trajectories aiming at teaching the four approaches of solving quadratic equations*: solving simple quadratic equations by square root method, null-factor law (simple factorization) method; solving general quadratic equations in types of $x^2 + px + q = 0, (p \neq 0, q \neq 0)$ and $ax^2 + bx + c = 0, (a \neq 0, b \neq 0, c \neq 0)$ by completing the square method and quadratic formula (*the pq-formula*). In the geometrical models (Model 1, 2, 3 and 4), it is possible to see how the parts (sides and small sub-areas of a rectangle or a square) and the whole (the entire area of a rectangle or a square) are connected. The embedded PCK is to concretise the related algebraic operational laws and make rich connections among them, showing also the part-whole relationship of factors and product in algebraic expressions.

Although unit 6 of factorization is not sequenced by the part-whole relationship, factorization is based on the first five units and contains the reversed procedure. The content of factorization can be regarded as an extension of the first five units. Therefore, it is indirectly connected to the introduction activity which seems to be a preview for these five units and the factorization. The first six units in the textbook have built up a basic ground in a progression in which teaching solving quadratic equations (including unit 7, 8 and 9) is the final goal. In this progression, the five embedded teaching trajectories are involved. The algebra content illustrated by the four varied geometrical models expands from the basic algebra laws or rules to an algebraic context: solving quadratic equations.

Unit 7, 8 and 9

Solving quadratic equations is a central topic in Unit 7, 8 and 9. These three units are sequenced from solving simple quadratic equations to solving general quadratic equations of the types $x^2 + px + q = 0$ or $ax^2 + bx + c = 0$. Presentations of the four methods for solving quadratic equations in these three units depend heavily on the basic procedural knowledge of computing polynomials in the first six units. Among these four methods, the square root method and the method of completing the square are based on the use of square rules while the method of null-factor law is based on the use of factorization. The fourth method of using quadratic formula or the pq-formula is derived from the method of completing the square. If the first six units would not have been presented or taught, it will be impossible to present or teach these four solving methods directly. Therefore, *the first six units can be seen as the basic teaching trajectories that set a ground for these three units in the whole teaching trajectory for solving quadratic equations*. Thus, the content knowledge (CK) of solving quadratic equations consists of the first six units as a base and the later three units as a central topic. *The aspect of PCK here is the organization of these units in a cumulative order to form a progressive sequence that can be regarded as a hypothetical overall teaching trajectory*.

The movement from polynomials to quadratic equations could raise questions such as: what is the difference between a second degree polynomial and a quadratic equation? Where does the equal sign and zero come from? There is no answer explicitly given in the textbook. *The transition from second degree polynomials to quadratic equations is not explicitly presented in the textbook* except when an example of growing bacteria expressed by a quadratic function in the beginning of Unit 7 is used. The example is described below:

> The number of bacteria y in a bacteria culture can be calculated with a second degree polynomial in x minutes after the experiment starts: $y = 25x^2 + 350x + 2500$. How long does it take for the number of bacteria to increase to 10000? We get the answer from the quadratic equation: $25x^2 + 350x + 2500 = 1000$ that is $25x^2 + 350x - 7500 = 0$. In general, a quadratic equation can be written as $ax^2 + bx + c = 0$ where a, b, and c are constants and $a \neq 0$ (Alfredsson et al., 2007, p. 24).

The last sentence in this quoted paragraph mentions the form of *general quadratic equations* but does not tell the difference between a second degree polynomial and a quadratic equation. However, this transition could be illustrated with the help of the geometrical models (Model 2 and 4) used in Unit 8 when the method of completing the square was presented. The completing the square method in Unit 8 is illustrated geometrically through an example of solving a quadratic equation $x^2 + 6x - 16 = 0$. Before completing the square, the equation needs to rewrite into $x^2 + 6x = 16$, which means that the whole area of the rectangle has the value of 16. The three geometrical models represent three steps of completing the square (Figure 19). Step one: a rectangle model (Model 2) represented by $x^2 + 6x$ consists of a square x^2 and a rectangle $6x$ with one side as 6; Step two: the same rectangle $x^2 + 6x$ becomes the square x^2 with two equally divided rectangles and one of the divided rectangle $3x$ is moved to another side of the square. This model represents an algebra expression $x^2 + 2 \cdot 3x$; Step three: a new square (the bigger one) is completed (Model 4) through adding a small square 3^2 in the right corner of the middle model. The completed new square is obtained: $x^2 + 6x + 3^2 = (x + 3)^2$. The little shaded square represents the constant term of $(6/2)^2$ or 3^2. After completing the square, the equation can be written as: $x^2 + 6x + 3^2 = 3^2 + 16$ and the value of the bigger area is 25. The roots of the equation are 2 and -8. This method is called completing the square.

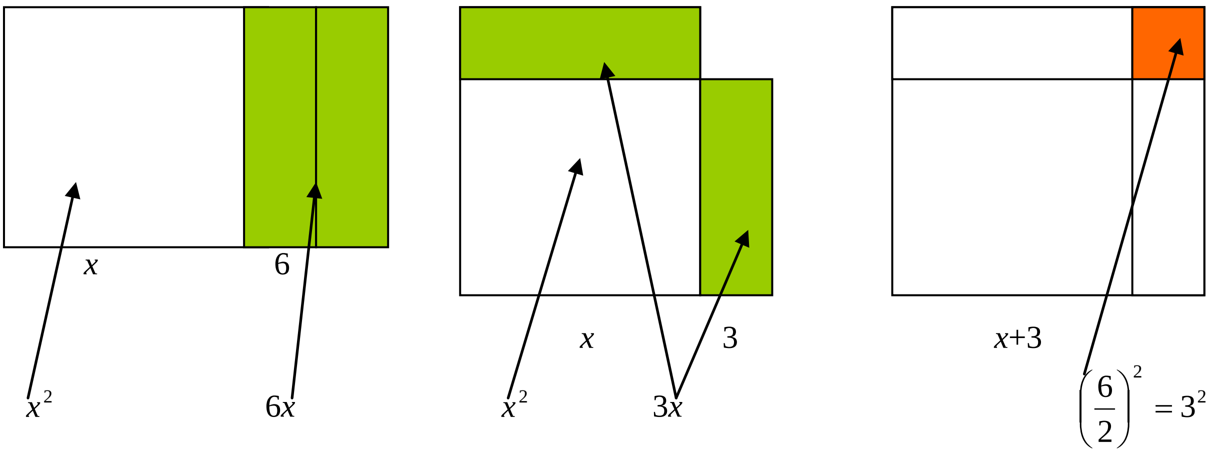


Figure 19. The three geometrical models used to illustrate procedures of completing the square method in *Matematik 4000 B* (Alfredsson et al., 2007, p. 26).

Among these three models, the first two are mathematically equivalent since they both represent the same area namely 16 in this example. The last model of a completed square changes the area by adding a small square in the right corner of the middle model. The essential concept here is that the equation $x^2 + 6x = 16$ is constructed when the area of the rectangle $x^2 + 6x$ (a polynomial) is given to be 16. The transition between a second degree polynomial and a quadratic equation could be explained here. But this explanation is absent from the textbook. These three geometrical models are used to illustrate the process of completing a square from a rectangle. The models used in the introduction activity have laid a foundation for teaching the procedure of completing the square but these three models do not give an explicit explanation as to what the value of 16 refers to.

Similarly, the three geometrical models in Unit 8 have another version in algebra history through putting the first geometrical model (Model 2) and last one (Model 4) together into one model according to the ancient Greek mathematician Euclid's geometrical solving method referred to in a proposition in his work *Data* (Katz & Barton, 2007). Euclid's geometrical figure showed both the equivalent relationship of a quadratic equation and the solving method referring to the quadratic formula. The following figure is the combination of Model 2 and 4 in the Euclid's geometrical proposition. The details can be found in the previous Chapter 3.1 on algebra history.

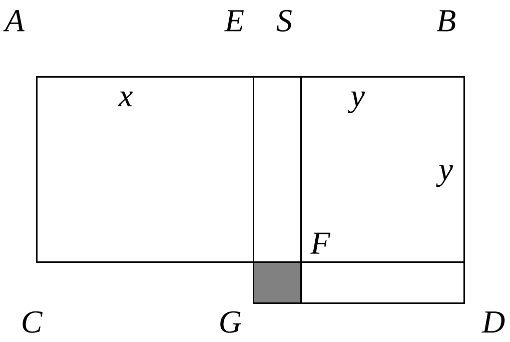


Figure 20. Euclid's geometrical figure was used for solving quadratic equations (Katz & Barton, 2007, p. 189).

In Euclid's model, rectangle *ACFS* is the same as Model 2 or similar to the first model in Unit 8 and the square *EGDB* is the same as Model 4 representing the completed square in Unit 8. The little square in grey colour is like the added square 3^2 in Unit 8 (Figure 19). It is possible that Euclid's geometrical figure or model could be used in Unit 8 and even Unit 9 to show how the quadratic formula is generated. But Unit 9 in this textbook is totally based on completing the square method illustrated in Unit 8 and presents pure algebraic procedures without any assistance of geometrical models as follows:

An example of a quadratic equation

$$x^2 + 5x + 6 = 0$$

$$x^2 + 5x = -6$$

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2 - 6$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{25}{4} - 6$$

$$x + \frac{5}{2} = \pm\sqrt{\frac{25}{4} - 6}$$

The generalized quadratic equation

$$x^2 + px + q = 0$$

$$x^2 + px = -q$$

$$x^2 + px + \left(\frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$$

$$\left(x + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$$

$$x + \frac{p}{2} = \pm\sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x=-\frac{5}{2}\pm\sqrt{\frac{25}{4}-6}\left[=-\frac{5}{2}\pm\frac{1}{2}\right] \qquad x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

(Alfredsson et al., 2007, p. 28)

Through comparing with an example, the quadratic formula is deduced. This final goal in the whole progression is reached without the connection with geometrical models, although it could have been done by making use of Euclid's geometrical explanations. No part-whole relationship is made explicit in this second last phase of the teaching trajectory.

Unit 10

According to the analysis, the last unit (Unit 10) is about algebraic applications of Pythagoras theorem and quadratic equations in solving geometrical exercises and mathematical problems related to real world situations. The real world problems in this unit are more advanced than the exercises offered in the previous units. The whole teaching trajectory regarding quadratic equations ends with Unit 10 and reveals an intended pedagogical idea of applying mathematics to solve real world problems which mirrors the same idea mediated by the picture of an architecture building in the very beginning of the textbook before the introduction activity. The picture is marked with a short comment "Using algebra as a language, we can describe geometrical forms" (Alfredsson et al., 2007, p. 6).

Conclusion

To sum up the results derived from the fourth round of analyses concerning embedded trajectories, it has been found that *the mathematical content presented in the ten units are constructed and organized in an overall teaching trajectory including five embedded teaching trajectories linked to the four geometrical models from the introduction activity in the beginning of the algebra chapter in the textbook*. The application of the geometrical models makes algebra content comprehensible and bridges the geometrical images to algebra representations. At the same time, they also help to structure the overall teaching trajectory aimed at solving quadratic equations by the quadratic formula (*the pq-formula*). The five embedded teaching trajectories play an important role in developing the mathematical content in the textbook in an overall teaching trajectory from a basic level (referring to the basic algebra operational rules and first three solving methods) to an abstract level (referring to the quadratic formula and algebra applications). These geometrical models have *their roots in algebra history* (Euclid's geometry method for solving quadratic equations in relation to the quadratic formula) and they represent knowledge of algebra regarding solutions of quadratic equations. In this sense, they are used as artefacts by the textbook. They also have potential to illustrate the abstract content of factorization and the quadratic formula. *The multi-functions of the geometrical models* imply how useful and powerful they can be if they are used in teaching quadratic expressions and equations as an alternative approach. Such embedded PCK is discovered from the analysis of this textbook.

The analysis has also revealed an essential pedagogical idea of applying mathematical rules and expressions as tools to solve real life situation problems or pure mathematics problems such as numerical computation and geometry. This pedagogical idea is exposed in the picture of an architecture building in the beginning of the book and in some real world problems of the last unit (Unit 10) on algebraic applications. In such a way, the PCK aspect of applying mathematics to contextualised problems from the real world or from the area of mathematics, opens and closes the whole progression with another embedded overall teaching trajectory as a circle (see Figure 21).

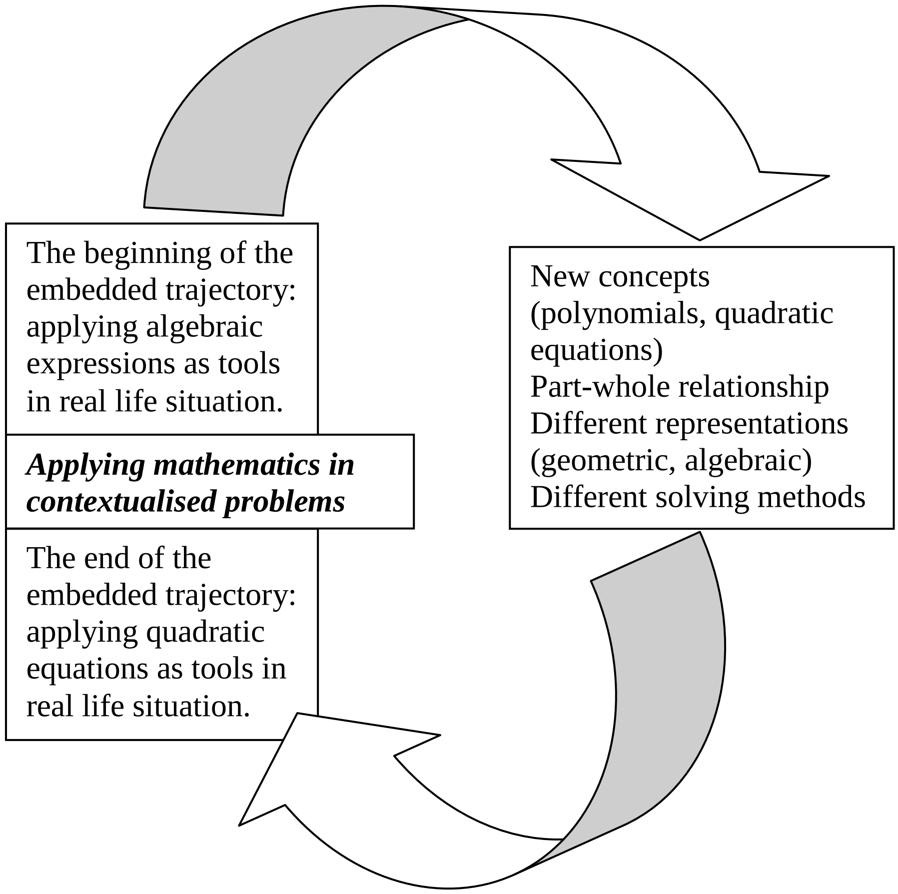


Figure 21. Another embedded teaching trajectory relates to the application of mathematics as tools for solving problems about real world situations and about pure mathematics.

This pedagogical idea of applying mathematics to real life situations has pervaded in a few examples and exercises in this textbook (Units 1, 2, 7 and 10), *but most of the units are still dominated by presenting mathematics procedural knowledge isolated from real world contexts*. It can be deduced by this analysis that the mathematics presented in this textbook reflects *a mixed version of formal mathematics such as algebra manipulations and rules* (Jakobsson-Åhl, 2006) *and modern version of mathematics applications and modeling* (De Lange, 1996; Freudenthal, 1991) *as well as algebra reasoning and generalization* (Kieran, 2007).

6.3 Summary of the findings

The content analysis in this study has been carried out in four rounds of analyses. The first and third rounds of analyses are quantitative analyses while the second and fourth are qualitative ones. The fourth round of analysis is a deep analysis based on the results derived from the previous rounds. Without the quantitative analyses, it would not have been possible to make a decision for the last round of analysis to focus on one textbook.

The first round of analyses obtains a general finding on what algebra content related to quadratic equations that is presented in all the eight textbooks. The general finding is that all the algebra content from these textbooks is very similar and the four solving methods with the focuses on completing the square method and the quadratic formula are presented in every textbook. The factorization method is presented but not for solving the general quadratic equations as presented in Chapter 3.2.2b). The finding indicates that the mathematics textbooks follow the Swedish mathematics syllabus.

The second round of analyses finds that the related algebra content regarding quadratic equations is presented in a cumulative order so that every topic builds on the previous topic. The absence of the factorization method for the general quadratic equations is furthermore proved in the second round of the analyses. It is also found that the absent factorization method is often more effective in solving quadratic equations than the quadratic formula since it requires fewer operational steps than the quadratic formula method does. In order to find multiple ways of presenting quadratic equations, four more textbooks are added to the eight ones in the third round of analyses.

The third round of analyses finds that the graphical approach to presenting quadratic equations is not so common, but applying algebraic approach for quadratic functions is common. Quadratic equations and functions are treated in separate chapters and lack detailed definitions even though it would be worth comparing them in order to avoid conceptual confusion. Presentation of quadratic equations often comes, in order, before quadratic functions. This implies that it is necessary to put emphasis on analyzing the presentation of quadratic equations only since representation of quadratic function does not have much influence on quadratic equations in most of the textbooks.

The results derived from the previous analyses have focused on looking for the content knowledge (CK) related to the subject of quadratic equations. The findings make the fourth round of analyses–on one textbook–aim at finding the embedded pedagogical content knowledge (PCK). The fourth round of analyses finds that the presentation of quadratic equations is constructed by an embedded overall teaching trajectory consisting of five trajectories based on the four algebraic historically related geometrical models. This indicates that it is important to prepare the students with basic algebra knowledge before they meet the four approaches of solving quadratic equations, in particular the abstract quadratic formula. The embedded PCK is revealed by the findings of the teaching trajectories and the powerful function of geometrical models as well as algebra application related to real world problems. With the help of the CK-PCK analytical tools, mathematical activities provided by the textbook reveal the pedagogical idea of “learning by doing” in mathematics, even if it is in the abstract algebra context. The provided mathematics exercises facilitate opportunities for the students to practice the newly learned algebra knowledge at different levels. The exercises cover different cognitive areas: basic algebraic procedural and conceptual training; reversed operation procedures; algebra structure sense; mathematical proofs; application of algebra; relational understanding of variables and parameters of quadratic equations as well as algebraic operational rules.

7. Discussion and Conclusion

The aims of this study are to investigate the algebra content related to quadratic equations and to find the embedded teaching trajectories related to quadratic equations. This study starts with my wondering why factorization is not the focus on teaching how to solve quadratic equations in Swedish upper secondary school as compared to my Chinese educational background. Seeking the answer to this question involves two areas: mathematics content of algebra and teaching algebra. The decision to analyze Swedish mathematics textbooks at upper secondary level was made because of their essential role in mathematics teaching (e.g. Johansson, 2006) and that they not only contain algebra content as the subject matter knowledge or CK (Mishra & Koehler, 2006) but also the intended pedagogical content knowledge (Shulman, 1986b) including different ways of representing and formulating algebra content to make it comprehensible to others. The plural aspects of this study make it necessary to perform extensive literature studies in the areas of pedagogical content knowledge from a teaching perspective, artifacts theory (Wartofsky, 1979), mathematics applications perspective (De Lange, 1996; Goldin, 2008; Vergnaud, 1987), mathematics textbook analytical framework (Brändström, 2005; Pepin et al., 2001; Schmidt et al., 1997; Van Dormolen, 1986), the history of algebra (Derbyshire, 2006; Kvasz, 2006), and factorization in algebra from mathematics as a discipline perspective (Durbin, 1992; Vretblad, 2000). The literature studies within the different areas have paved the way for a background for the content analytical criteria in this study and for finding content links with algebra history through looking at the historical background of algebra development. The literature review work on previous research also takes the plural aspects of this study into account, thus it includes a review of previous studies on textbook analyses including mathematics textbooks and a review of teaching and learning algebra related to different approaches for solving quadratic equations.

The results of the literature reviews show that content analysis is dominant in the field of textbook research (Johnsen, 1993). Studies related to PCK theoretical perspectives have mostly been carried out through classroom observations and interviewing teachers. Little research has been carried out on relating CK-PCK aspects to the content analysis (Johansson, 2006; Pepin et al., 2001). In the field of teaching and learning school algebra, few studies have been related to quadratic equations (Kieran, 2007). These findings from the literature studies and research reviews have been helpful when designing this study on mathematics textbook analyses, making this study focus on content analysis aiming at exploring the embedded pedagogical content knowledge through examining algebra content related quadratic equations and factorization in the mathematics textbooks.

In this chapter, the results will be discussed in relation to the two research questions asked in the beginning of the thesis. Practical implications will be put forward and the limitation of the study will be pointed out. In the end, I will give suggestions for future studies.

7.1 Discussion of the results

Two research questions have been asked in this study:

1. What mathematics do Swedish upper secondary mathematics textbooks reflect in their presentations of quadratic equations?

2. What aspects of pedagogical content knowledge can be traced in the way a Swedish upper secondary school textbook presents the algebra content of quadratic equations?

Considering these two research questions, the discussion of the results focuses on the following aspects: algebra content knowledge; factorization and quadratic formula; the PCK aspects containing historically related geometrical models; provided exercises; embedded trajectories; and CK-PCK framework.

Algebra content knowledge

The results derived from the quantitative analyses of twelve mathematics textbooks show that mathematics content related to quadratic equations in the textbooks is similar in all books. This reflects the algebra content goals for the B course in the Swedish mathematics syllabus (Skolverket, 2000). This result is the same as the one derived from previous studies that mathematics textbooks cover the same topics as in the curriculum (Johansson, 2006; Venezky, 1992; Pepin et al., 2001). In all of the investigated textbooks, algebra content related to quadratic equations is presented like algebra knowledge package (Ma, 1999). In this knowledge package, pre-knowledge consists of: *polynomial simplification, distributive laws, the difference-of-two squares law, square rules and factoring quadratic expressions* to serve as a base for the presentations of the four solving methods *the square root method*, *the factorization method*, *completing the square method*, and *the quadratic formula*. In order to approach the presentation of the four solving methods, the pre-knowledge is needed. All the algebra content elements in this knowledge package regarded as subject matter content knowledge (CK) have built up an embedded overall teaching trajectory in accumulative relationship to reach a final goal of solving quadratic equations by the quadratic formula. Among these algebra content elements, the last two solving methods are emphasized. The factorization method is included but only applied for solving simple quadratic equations. Solving the general quadratic equations by the factorization method is absent from all the textbooks.

This knowledge package characterizes algebra manipulation. Thus, the algebra content related to quadratic equations in the investigated textbooks reflects rule-bound and convention-bound mathematics (Pepin et al., 2001) with a goal for encouraging procedural knowledge (Hiebert & Lefevre, 1986).

Factorization and quadratic formula

Among the four solving methods presented in the textbooks, the factorization method for solving general quadratic equations of the kind $ax^2 + bx + c = 0$, is absent. For example, $2x^2 - 5x + 2 = 0 \Leftrightarrow (2x-1)(x-2) = 0$. The findings from previous studies (Bossé & Nandakumar, 2005; Hoffman, 1976; Leong et al., 2010; Kemp, 2010; Kennedy & et al., 1991; Nataraj & Thomas, 2006; Zhu & Simon, 1987) indicate that this kind of factorization is a common topic in elementary algebra teaching in some other cultures. The absence of this kind of factorization from Swedish mathematics textbooks implies an answer to my wondering why factorization was not in the focus of teaching quadratic equations in a Swedish school. Mathematics textbooks influence mathematics teaching. Textbooks often define teaching aims and what teachers present mostly comes from textbooks (Englund, 1999). On the other hand, the result from a previous study shows that mathematics textbooks are designed and constructed differently depending on different pedagogical culture (Pepin et al., 2001). What is not presented in textbooks is probably not presented by teachers. The absence of this kind of factorization implies that Swedish upper secondary students may not get the chance to learn this kind of factorization since it is not presented in mathematics textbooks.

One of the essential approaches for solving quadratic equations according to my study is to use the quadratic formula $x_{1,2}$ = (-p/2) ± √[(p/2) − q^2]. The result shows that this the pq-formula is the intended final goal in the overall teaching trajectory. This result can probably explain a phenomenon in a previous study (Olteanu, 2007). The phenomenon of that study (Olteanu, 2007) shows that students have developed a stronger relationship to the the pq-formula and an equation but a weaker relationship between a function and an equation. The students have difficulties in seeing the equivalence between two different kinds of quadratic equations: $ax^2 + bx + c = 0$ ($a \neq 0$) and $x^2 + px + q = 0$. This may be because the mathematics textbooks only focus on using the pq-formula to solve quadratic equations of the type $x^2 + px + q = 0$. The possible consequence is that teachers and students would regard it as the important algebra content. The teachers' presentations of mathematical content are often portrayed in the textbook (Johansson, 2006). This means that the absence of the general quadratic equation $ax^2 + bx + c = 0$ in the textbook may make the teaching of algebra less focus on this kind of quadratic equation.

Another version of the pq-formula is $x_{1,2}$ = [-b ± √(b^2 − 4ac)]/(2a) then ($a \neq 0$). This version is mentioned in an exercise in this textbook but left without any emphasis. Olteanu (2007) asks why the teaching of solving quadratic equations does not apply the abc-formula directly since the abc-formula can be used directly to solve a quadratic equation without changing all the general quadratic equations into the type of $x^2 + px + q = 0$ equations. Furthermore, the abc-formula can facilitate the identification of the extreme point of a quadratic function. Since the textbooks do not emphasize this abc-formula, it is possible that teachers will not teach the abc-formula, or not put focus on it. What is taught in mathematics classrooms is influenced by what teaching material is used (Johansson, 2006).

Olteanu (2007) has found that students have difficulties in telling the difference between quadratic functions and quadratic equations. In response to her finding, my study has implied that a lack of formal detailed definitions for quadratic equations and functions may result in difficulties for the students in understanding the difference between a quadratic function and a quadratic equation.

The PCK aspects containing historically related geometrical models

The results derived from one textbook analysis show that the mathematics in that textbook has a mixed version of algebra history and mathematics application and manipulation. The idea related to algebra history in the textbook has influenced the presentation and organization of the algebra content concerning quadratic equations. In the textbook, the application of the four geometrical models related to algebra history has led to the five embedded teaching sub-trajectories in relation to the distributive law, the square rule, and completing the square method in a particular order with a part-whole relationship. The multi-functions of the geometrical models make them artifacts (Wartofsky, 1979) that are important algebraic representations related to historical roots at the same time as they embrace pedagogical content knowledge of transferring the knowledge of the algebraic rules, as shown in the analysis. In other algebra teaching culture, these geometrical models named as algebra tiles (Leong et al., 2010) are applied for illustrating factorization of quadratic expressions. The application of the historically related geometrical models or algebra historically related approach is an important aspect of pedagogical content knowledge (PCK) built in this textbook.

The result implies that the embedded teaching sub-trajectories with the geometrical models can, as alternative teaching instructions, contribute to classroom teaching related to quadratic equations. The embedded overall teaching trajectory can provide the whole progression of teaching how to solve quadratic equations with the quadratic formula as the final goal. The study suggests that the application of the geometrical models in relation to Euclid's geometrical method (Kvasz, 2006) for solving quadratic equations could be used to illustrate the quadratic formula in order to provide concrete visual explanations. On the other hand, the other algebra representations, like a graphical approach related to quadratic functions, is another alternative to solving quadratic equations, but more than half of the investigated textbooks have not made use of this approach.

Another aspect of the embedded PCK is about mathematics application in the provided mathematics exercises and activities in the textbook. Mathematics application (Freudenthal, 1991; Van Den Heuvel-Panhuizen, 2003) is reflected within two contexts: real world context and pure mathematics context. The real world context involves word problems in the exercises. They are often difficult since they are somewhat infrequent and have not been presented in the previous examples. The solving procedure of such problems includes first interpreting word problems into mathematical expressions and then finding solving methods. The PCK of using the real world problems intends to bring reality into the mathematics classroom and create opportunities for getting practical contact with the real world situation (Chapman, 2006). It makes students learn mathematics through doing mathematics, something which reflects the idea of mathematization (De Lange, 1996). This is the case of two activities of which one is about finding rules for numerical computations and the other is about finding the relationship between roots and coefficients of a quadratic equation. In these two activities, students are encouraged to work together and generalize the algebra rules and expressions through working with the exercises. The mathematical content is not of the kind real world problems but instead related to pure mathematics context. The activity process has the character of mathematization. However, algebra application in real world problems is not new but appeared in Swedish mathematics textbooks in the late 1970s (Jakobsson-Åhl, 2006).

Provided exercises

The algebra content textual presentations in the textbook take up little space compared to the provided exercises and activities. More than half of the mathematics exercises are routine tasks according to my findings. The other half are exercises aiming at practicing the structure of quadratic equations, reversed operation procedures, conceptual understanding of newly learned algebra rules and concepts. Exercises related to the application of algebra imply the usefulness of algebra in our daily life and the subject of mathematics. Extra exercises such as discussion activities, a homework exercise part, two mathematics tests equivalent to the national tests and *Problems for everybody* are provided at the end on the chapter of algebra. The level of difficulty increases in these extra exercises. These extra mathematical activities, homework, tests and problems at the end of the chapter are like a "smorgasbord" providing a large amount of various exercises for students to practice the algebra content presented in this chapter.

A large amount of exercises imply two pedagogical goals: acquisition of knowledge with activities and acquisition of process skills with content knowledge growth (Van Dormolen, 1986). This can also imply that teachers may use this textbook in terms of rich exercises in school and for homework to facilitate students' learning at different levels (Pepin et al., 2001). At the same time, students use the textbook to assist with self study. The result related to the large amount of mathematics exercises and the group activities in this textbook has supported an opinion that the function of textbook is not just for mediating the facts of subject matter knowledge but encouraging active learning through reconstructing the knowledge with a pedagogical idea of utilizing the subject matter knowledge as a process rather than amount of facts (Selander, 2003). At this point, the pedagogy of "learner-centered design (LCD)" (Selander, 2003, p. 217) is built into the design of the exercises and activities of this textbook. The shortcoming is the uncertainty of students choosing proper exercises by themselves and if they have enough time to do them. Teacher's guidance is needed.

The algebra content and the provided exercises in this book cover and reflect the goals expressed in the Swedish mathematics syllabus for course B: "Pupils should be able to interpret, simplify and reformulate expressions of the second degree, as well as solve quadratic equations and apply this knowledge in solving problems" (Skolverket, 2000). Mathematics at this level is about algebra generality, manipulations, structure and application with focus on algebraic operational rules and solving methods which belong to procedural knowledge (Hiebert & Carpenter, 2007; Hiebert & Lefevre, 1986).

The embedded teaching trajectories

The textbook reflects a relation between elementary algebra and geometry which in nature relates to mathematical history. The algebra history related to the PCK makes use of the geometrical models which originated from algebra history to present and organize algebra content according to a certain order. In such a way, the textbook provides the algebra content knowledge of quadratic equations by an embedded overall teaching trajectory consisting of five hypothetical teaching sub-trajectories related to quadratic equations. The relationship between the overall trajectory and five sub-trajectories is that these five sub-trajectories together build up a progression developed from presenting the basic algebra knowledge to the more complicated and abstract algebra knowledge with a final goal. The overall teaching trajectory covers all the sub-trajectories and ends with the final goal.

From a theoretical perspective, this study is an attempt to combine the CK concept (Mishra & Koehler, 2008) and the PCK concept (Shulman, 1986b) with text analysis criteria for analyzing the mathematics textbook. The CK-PCK framework functions as an overall analytical tool whose parts can not be separated from each other (Ferrini-Mundy et al., 2003). The subject matter content knowledge and pedagogical content knowledge reflected from the textbook have to be studied together in order to explore the embedded PCK in the textbook. The findings of this study are the results from all the four rounds of analyses. The first three rounds of analyses, in particular the first and third quantitative analyses, play an important role for the fourth round of in-depth content analysis. These four rounds of analyses have together shaped the whole study.

7.2 Practical implications

This study is about algebra content analyses within the area of quadratic equations presented in the mathematics textbooks used for mathematics course B at Swedish upper secondary school. As a teaching and learning resource, textbooks are used by both students and teachers. The results derived from this study have exposed the embedded teaching ideas and mathematical connections. For teachers, the historically related pedagogy reflected in this textbook may guide them in putting teaching emphasis on illustrating algebra rules by geometrical representations. The geometrical models in the textbook serve as artifacts and can become very useful and powerful in classroom teaching if teachers realize their functions and get familiar with the history of algebra. The five embedded teaching sub-trajectories and an overall trajectory can assist teachers in organizing the teaching of quadratic equations. The embedded trajectories may possibly be developed into a geometrically and historically related algebra teaching model in practice. The textbook has also provided the basic knowledge of solving quadratic equations. Therefore it gives both teachers and students an implication of what one needs to know before learning to solve quadratic equations.

The finding of the absence of factorization for general quadratic expressions may bring teachers' attention to develop alternative approaches to solving quadratic equations further. Thereby students could get more opportunities to develop their algebra competence for structure and number sense. In that case, teaching different solving methods is not the only goal, but it helps students get a deep insight into algebra structure at the same time as fostering their algebra thinking should be concerned for their future study of abstract algebra at an advanced level. On the other hand, "algebra is a large content area, too large to fit entirely within any one school curriculum, and so choices must be made" (Kendal & Stacey, 2004, p. 345).

The result of the analyses of various exercises in forms of different activities, problems, and tests may guide teachers when choosing proper activities and exercises according to the students' needs and the level they are on. The mathematics application pedagogy in mathematical context and activities as well as real world problems may inspire teaching algebra depending on what kind of mathematics that is emphasized in the classroom. The pedagogical content knowledge of offering different kinds of exercises and activities revealed by the analyses may be useful for teachers in order to organize and help students in their own learning.

For students, the textbook can be used as a guide book for their self study since there are plenty of various mathematics exercises at different levels. Students can choose the ones appropriate for themselves, if they know their own level.

For textbook writing, the study implies a need for formal algebra definitions of some algebra concepts and for more verbal explanations of contextual connections. It is also necessary to give clearer instructions of different kinds of exercises so that users know how to use them.

The framework used in the analyses may contribute to studying teachers' content knowledge of teaching algebra in the classroom.

7.3 Critical reflection on this study

The analyzing process in this study has included four rounds of analyses, which I performed until I found the embedded trajectories from one analyzed textbook. The study field was narrowed down from the twelve textbooks to one textbook. The first three rounds of analyses aimed at finding algebra content related to quadratic equations in general and looking for their presentation orders as well as different approaches to solve quadratic equations. In such a way, the first three rounds of analyses set up a base for the last analysis aiming at seeking the PCK aspects and how the embedded trajectories are presented. The results concerning teaching trajectories were derived from the final analysis and can not be used for generalizing all the investigated textbooks since this is a qualitative study. These embedded teaching trajectories are hypothesized (Cobb, 2001). The finding is a suggestion only for how teaching algebra related quadratic equations could be done.

All the analysis data for this content analysis is described according to my own interpretation based on my experiences and knowledge of algebra and teaching. The analyses might possibly be influenced by my own background, in particular my Chinese educational experience in mathematics learning. The factorization method is an example of it. I regard it as an effective method to solve quadratic equations, but this might not necessarily be the case for other mathematics teachers. The embedded pedagogical content knowledge is uncovered according to my own teaching and learning experience. Take exercise 1228 (Alfredsson et al., 2007, p. 30) as an example:

> Louis and Nille want to solve the equation $x^2 + x - 2 = 0$ with the quadratic formula.
> Louis: "One of the coefficients of *x* is missing, *p* is 0."
> Nille: "We have one *x*, *p* is 1."
> Who is right? Solve the equation.

The embedded PCK is the knowledge of knowing students' misconception of $x = 0x$ which often is the case. Therefore, this task has as a purpose to check students' concept of the coefficient when the value is one which never is written before an unknown or a variable. The PCK in this task is interpreted according to my understanding.

What I have examined in this study is the content provided in the mathematics textbook. When Shulman (1986b) declared the PCK concept, he meant teachers' pedagogical content knowledge which is constructed by relating to the teacher (Emanuelsson, 2001). The PCK related research has very different focuses such as teaching, teachers' knowledge, content understood by teachers, content understood by students, content provided in teaching materials and so on (Ball et al., 2008; Emanuelsson, 2001; Ferrini-Mundy et al., 2003). How the analysis of the empirical material in such research is done depends on the researchers' interpretation of the PCK aspects. The core term in my study is content which consists of algebra as disciplinary content and pedagogical content knowledge of teaching algebra. Although this study does not involve classroom teaching content, it directly relates to the content provided in the mathematics textbook. Agreeing with Emanuelsson (2001), I regard it as meaningful to view all content as having pedagogical dimensions as long as they are used for educational purposes though the content is from the textbook, not from the classroom teaching.

7.4 Suggestions for future studies

This study is directed at upper secondary algebra teaching in the special area of quadratic equations. The theoretical framework in this study is CK-PCK (Mishra & Koehler, 2008; Shulman, 1986b). This study attempts to find algebra content knowledge related to quadratic equations as a subject and explore the pedagogical content knowledge of teaching how to solve quadratic equations. The result of the embedded teaching trajectories and the overall trajectory is tentative since it is derived from the analysis of the textbook.

It would be interesting to see the result of this study in practically use. How are these trajectories and the overall trajectory used in classroom teaching when the same textbook is used? The next step in a future study would be to find out how these teaching trajectories and the overall trajectory are manifested in the teaching of quadratic equations in algebra classrooms. This is a natural step in continuing this actual study. A continued study would focus on the application of these trajectories for teaching. The aim would be to find the effectiveness of these geometrically and historically related teaching trajectories. How do these trajectories influence both teaching and learning related to a special subject on solving quadratic equations? The research method would involve classroom observations, interviewing teachers and students and probably tests. In such a way, the pedagogical content knowledge of teaching quadratic equations would be deepened in a continuing study.

Another suggestion for a continuing study is to widely analyze the algebra content related to quadratic equations, and even quadratic functions, with a CK-PCK framework in more mathematics textbooks from both Sweden and other countries. In this case, a comparative study would be meaningful. The aim would be to reveal embedded pedagogical content knowledge from other cultures and enrich Swedish mathematics textbook writing and algebra teaching. At the same time, the CK-PCK framework and the analytical criteria derived from this study will be developed by a future study.

References

Abell, S. K. (2007). Research on Science Teacher Knowledge. In S. K. Abell., & N.G. Lederman (Eds.), *Handbook of Research on Science Education* (pp.1105-1149). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers, cop.

Alfredsson, L., Brolin, H., Erixon, P., Heikne, H., & Ristamäki, A. (2007). *Matematik 4000: Kurs B blå lärobok*. Stockholm: Natur och Kultur.

Alfredsson, L., Erixon, P., & Heikne, H. (2008a). *Matematik 4000 Kurs A: Lärarhandledning till kurs A röd och kurs A grön*. Stockholm: Natur och kultur.

Alfredsson, L., Erixon, P., & Heikne, H. (2008b). *Matematik 4000: kurs B röd lärobok*. Stockholm: Natur och Kultur.

Allaire, P. R., & Bradley, R. E. (2001). Geometric approaches to quadratic equations from other times and places. *Mathematics Teacher, 94*(4), 308-319.

Anderson, L. W., & Krathwohl, D. R (Eds.). (2001). A Taxonomy for learning, teaching, and assessing: A Revision of Bloom's taxonomy of educational objectives. New York: Longman.

Asquith, P., Stephens, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts equal sign and variable. *Mathematical Thinking and Learning, 9*(3), 249-272.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning*. Westport, CT : Ablex Publishing.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Berry, J., Fentem, R., Partanen, A.-M., & Tiihala, S. (2004). The use of symbolic algebra in learning mathematics: The barrier from formal examination schemes. *Nordisk Matematikk Didaktikk, 9*(4), 49-64.

Björk, L.-E., Borg, K., Brolin, H., Ekstig, K., Heikne, H., & Larsson, K. (2000). *Matematik 3000: Matematik tretusen kurs B.* Stockholm: Natur och Kultur.

Björup, K., Körner, S., Oscarsson, E., & Sandhall, Å. (2000). *Nya Delta: Matematik kurs A och B*. Malmö: Gleerups.

Bloom, B. S., & Krathwohl, D. R. (1956). *Taxonomy of educational objectives: The classification of educational goals. Handbook I: Cognitive domain*. New York: David McKay, cop.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects - State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68.

Boesen, J. (2006). *Assessing mathematical creativity: Comparing national and teacher-made tests, explaining differences and examining impact*. Umeå: Department of Mathematics and Mathematical Statistics, Umeå Univ.

Bossé, M. J., & Nandakumar, N. R. (2005). The factorability of quadratics: Motivation for more techniques. *Teaching Mathematics and its Applications: An International Journal of the IMA, 24*(4), 143-153.

Brolin, H., & Heikne, H. (2003). *Matematik 3000: Matematik tretusen kurs B grundbok*: Stockholm: Natur och Kultur.

Brändström, A. (2005). *Differentiated tasks in mathematics textbooks: A analysis of the levels of difficulty*. Luleå : Luleå tekniska universitet.

Chapman, O. (2006). Classroom practices for context of mathematics word problems. *Educational Studies in Mathematics, 62*(2), 211-230.

Chiappetta, E. L., & Fillman, D. A. (2007). Analysis of five high school biology textbooks used in the United States for inclusion of the nature of science. *International Journal of Science Education, 29*(15), 1847-1868.
Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.
Cobb, P. (2001). Supporting the improvement of learning and teaching in social and institutional context. In S. M. Carver & D. Klahr (Eds.), *Cognition and instruction: Twenty-five years of progress* (pp. 455-478). New Jersey: Lawrence Erlbaum Associates, Inc.
Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education* (6^{th} ed.). London and New York: Routledge.
Danielsson, R., Gabrielsson, G., & Löfstrand, B. (2002). *Räkna med vux: Kurs B*. Malmö: Gleerups.
De Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 49-97). Dordrecht / Boston / London: Kluwer Academic Publishers.
Denzin, N. K., & Lincoln, Y. S. (2000). Introduction: The discipline and practice of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 1-28). Thousand Oaks, CA: Sage.
Derbyshire, J. (2006). *Unknown quantity*. Washington, D.C.: Joseph Henry Press.
Doyle, W. (1992). Curriculum and pedagogy. In P. W. Jackson (Ed.), *Handbook of research on curriculum: A project of the American Educational Research Association* (pp. 486-516). Toronto: Maxwell Macmillan Canada.
Durbin, J. R. (1992). *Modern algebra: An introduction*. New York, Chichester, Brisbane, Toronto and Singapore: John Wiley & Sons, Inc.
Emanuelsson, J. (2001). *En fråga om frågor: hur lärares frågor i klassrummet gör det möjligt att få reda på elevernas sätt att förstå det som undervisningen behandlar i matematik och naturvetenskap*. Göteborg, Sweden: Acta Universits Gothoburgensis.
Englund, B. (1999). Lärobokskunskap, styrning och elevinflytande. *Pedagogisk forskning i Sverige (Pedagogical research in Sweden) 1999, Årg 4 nr 4*, 327-348.
Ferrini-Mundy, J., Floden, R., & McCrory, R. (2003). *Knowledge for teaching school algebra: Challenges in developing an analytic framework*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.
Floden, R. E. (2001). Research on effects of teaching: A continuing model for research on teaching. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 3-16). Washington, D.C.: American Educational Research Association.
Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht, NL: Kluwer Academic Publishers.
Gagné, R. M. (1977). *The conditions of learning* (3rd ed.). New York etc: Holt, Rinehart and Winston.
Gennow, S., Gustafsson, I.-M., & Silborn, B. (2005a). *Exponent B: Matematik för gymnasieskolan [gul]* (2 ed.). Malmö: Gleerups Utbildning AB.
Gennow, S., Gustafsson, I.-M., & Silborn, B. (2005b). *Exponent B: Matematik för gymnasieskolan [red]* (2 ed.): Gleerups Utbildning AB.
Goldin, G. (2008). Perspectives on representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education* (2 ed., pp. 176-201). New York: Routledge.
Great Source Education Group. (2000). *Algebra to go. A mathematics handbook.* Wilmington, Massachusetts: A Houghton Mifflin Company.

Grepstad, O. (1997). *Det litterære skattkammer. Sakprosaens teori og retorikk*. Oslo: Det Norske Samlaget.
Gustafsson, C. (1980). *Läromedlens funktion i undervisningen*. En rapport från utredningen om läromedelsmarknaden (Ds U 1980:4). Stockholm: Utbildningsdepartementet.
Haladyna, T. (1997). *Writing test items to evaluate higher order thinking*. Needham Heights, MA: Allyn and Bacon.
Hamdan, M. (2006). Equivalent structures on sets: Equivalence classes, partitions and fiber structures of functions. *Educational Studies in Mathematics, 62*(2), 127-147.
Hiebert, J., & Carpenter, T. P. (2007). Learning and teaching with understanding. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65-97). Charlotte, NC: Information Age Pub.
Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.
Hoffman, N. (1976). Factorisation of quadratics. *Mathematics Teaching, 76*, 54-55.
Holmström, M. (2001). *Matematik från A till E: För komux och gymnasieskolan. Gymnasiets matematik kurs B*. Stockholm: Liber AB.
Hough, S., O'Rode, N., Terman, N., & Weissglass, J. (2007). Using concept maps to assess change in teachers' understanding of algebra: A respectful approach. *Journal of Mathematics Teacher Education, 10*(1), 23-41.
Howden, H. (1985). *Algebra tiles for the overhead projector: For Grade 6*. Illinois: Learning Resources.
Hsieh, H.-F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277-1288.
Häggström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Göteborg, Sweden: Acta Universitatis Gothoburgensis.
Jackman, T. (2005). Another method of factorizing quadratics. *Mathematics in School, 34*(3), 20-21.
Jakobsson-Åhl, T. (2006). *Algebra in upper-secondary mathematics: A study of a selection of textbooks used in the years 1960-2000 in Sweden*. Luleå: Luleå tekniska universitet/Matematik.
Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *The Journal of Mathematical Behavior, 23* (4), 371-388.
Johansson, M. (2006). *Teaching mathematics with textbooks: A classroom and curricular perspective*. Luleå : Luleå University of Technology.
Johnsen, E. B. (1993). *Textbooks in the kaleidoscope: A critical survey of literature and research on educational texts*. Oslo : Scandinavian U.P, cop.
Johnsson Harrie, A. (2009). *Staten och läromedlen: En studie av den svenska statliga förhandsgranskningen av läromedel 1938-1991*. Linköping: Institutionen för beteendevetenskap och lärande, Linköpings universitet.
Julin Svensson, A.-C. (2000). *Nya redskap för lärande: Studier av lärares val och användning av läromedel i gymnasieskolan*. Stockholm: HLS.
Kang, W., & Kilpatrick, J. (1992). Didactic transposition in mathematics textbooks. *For the Learning of Mathematics, 12*(19), 2-7.
Katz & Barton, V. J., & Barton, B. (2007). Stages in the history of algebra with implications for teaching. *Educational Studies in Mathematics, 66*(2), 185-201.
Kemp, A. (2010). Factorizing quadratics. *Mathematics in School: for secondary and college teachers of mathematics, 39* (4), 44-45.

Kendal, M., & Stacey, K. (2004). Algebra: A world of difference. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra* (pp. 390-419). Dordrecht: Kluwer Academic.

Kennedy, P. A., & et al. (1991). Factoring by Grouping: Making the connection. *Mathematics and Computer Education, 25*(2), 118-123.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707-762). Charlotte, NC: Information Age Pub.

Kilhamn, C. (2011). *Making sense of negative numbers*. Göteborg, Sweden: Acta Universitatis Gothoburgensis.

Kilpatrick, J. (1993). Beyond face value: Assessing research in mathematics education. In G. Nissen & M. Blomhoej (Eds.), *Criteria for scientific quality and relevance in the didactices of mathematics*. Roskilde: Roskilde University, IMFUFA.

Kotsopoulos, D. (2007). Unraveling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher, 63*(2), 19-24.

Krathwohl, D. R. (2001). A revision of Bloom's taxonomy: An overview. *Theory into Learning, 41*(4), 212-218.

Krippendorff, K. (2004). *Content analysis: An Introduction to its Methodology* (2nd ed.). Thousand Oaks, CA: Sage.

Kvasz, L. (2006). The history of algebra and the development of the form of its language *Philosophia Mathematica III 14*(3), 287-317.

Leong, Y. H., Yap, S. F., Yvoone, T. M., Mohd Zaini, I. K. B., Chiew, Q. E., Tan, K. L. K., & Subramaniam, T. (2010). Concretising factorisation of quadratic expressions. *The Australian Mathematics Teacher, 66* (3), 19-24.

Lester, F. K., & Lambdin, D. V. (1998). The ship of Theseus and other metaphors for thinking about what we value in mathematics education research. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Vol. 2, pp. 415 - 426). Dortrecht, The Netherlands: Kluwer Academic Publishers.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Education Studies in Mathematics, 30*(1), 39-65.

Lingefjärd, T. (2000). *Mathematical modeling by prospective teachers using technology*. Athens: University of Georgia.

Love, E., & Pimm, D. (1996). 'This is so': a text on texts. In A. J. Bishop (Ed.), *International handbook of mathematics education*. Dortrecht, The Netherlands: Kluwer Academic Publishers.

Långström, S. (1997). *Författarröst och lärobokstradition: En historiedidaktisk studie*: Umeå: Univ.

MacDonald, T. H. (1986). Problems in presenting quadratics as a unifying topic. *Australian mathematics teacher, 42*(3), 20.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Erlbaum.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054.

Nataraj, M. S., & Thomas, M. O. J. (2006). Expansion of binomials and factorisation of quadratic expressions: Exploring a Vedic method. *Australian Senior Mathematics Journal, 20*(2), 8-17.

Norberg, A., Viklund, G., & Burström, L. (2002). *Matematik B*. Stockholm: Bonnier utbildning.

Norton, S. J. (2007). Linking number and algebra through the distributive law. In J. Vincent, J. Dowsey & R. Pierce (Eds.), *Proceedings of the 44th annual conference of the Mathematical Association of Victoria* (pp. 233-250). University of Melbourne.
Obermeyer, D. D. (1982). Another look at the quadratic formula. *Mathematics Teacher, 75*(2), 146-152.
Olteanu, C. (2007). *"Vad skulle x kunna vara": Andragradsekvation och andragradsfunktion som objekt för lärande*. Kristianstad: Institutionen för beteendevetenskap, Högskolan i Kristianstad.
Pepin, B., Haggarty, L., & Keynes, M. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *Zentralblatt für Didaktik der Mathematik, 33*(5), 158-175.
Persson, P.-E. (2010). *Räkna med bokstäver: En longitudinell studie av vägar till en förbättrad algebraundervisning på gymnasienivå*. Luleå: Luleå tekniska universitet.
Rauff, J. V. (1994). Constructivism, factoring, and beliefs. *School Science and Mathematics, 94*(8), 421.
Reints, A. (1997). *Towards a model for analysis of textbook*. Paper presented at the International Associations for Research on Textbooks and Educational Media - The Fourth International Conference in Oslo.
Schmidt, W. H., McKnight, C. C., Valverde, G. A., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims - volume 1: A cross - national investigation of curricular intentions in school mathematics*. Boston: Kluwer Academic Publishers.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.
Selander, S. (2003). Pedagogiska texter och andra artefakter för kunskap och kommunikation: En översikt över läromedel - perspektiv och forskning In *Läromedel - specifikt: Betänkande om läromedel för funktionshindrade (SOU, 2003:15)*. Stockholm: Fritzes offentliga publikationer.
Shulman, L. S. (1986a). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of research on teaching: a project of the educational research association* (pp. 3-36). New York: Macmillan Publishing Company.
Shulman, L. S. (1986b). Those who understand: Knowledge growth in teaching *Educational Research, 15* (2), 4-14.
Sierpinska, A. (1993). Criteria for scientific quality and relevance in the didactics of mathematics. In G. Nissen & M. Blomhoej (Eds.), *Criteria for scientific quality and relevance in the didactics of mathematics*. Roskilde: Roskilde University, IMFUFA.
Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77, 20-26*.
Skolverket. (2000). Kursplaner och betygskriterier: Gymnasieskolan: Kursplan för MA 1202-Matematik B. Retrieved April 5, 2010. from http://www3.skolverket.se/ki03/front.aspx?sprak=SV&ar=0910&infotyp=5&skolform=21&id=3209&extraId
Smith, J. C. (2006). Revisiting algebra in a number theoretical setting. In R. Zazkis & S. R. Campbell (Eds.), *Number theory in mathematics education: Perspectives and prospects*. Mahwah, N.J.: Lawrence Erlbaum Associates, Publishers.
Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*(5), 344-350.
Stover, D. W. (1978). *Teaching quadratic problem solving*. Mathematics Teacher, *71*(1), 13-16.
Stray, C. (1994). Paradigms regained: Towards a historical sociology of the textbook. *Journal of Curriculum Studies, 26(1)*, 1-29.

Stray, C. (1997). *Paradigms lost: towards a historical sociology of the textbook*. Paper presented at the International Associations for Research on Textbooks and Educational Media - The Fourth International Conference in Oslo.

Szabo, A., Larson, N., Viklund, G., & Marklund, M. (2008). *Origo: Matematik kurs B för samhällsvetenskapliga och estetiska program*. Stockholm : Bonnier utbildning.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal, 18*(1), 47.

Wallin, H., Lithner, J., Wiklund, S., & Jacobsson, S. (2000). *NT/a+b: Gymnasiematematik för naturvetenskaps- och teknikprogrammen, kurs A och B*. Stockholm: Liber AB.

Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9-35.

Van Dooren, W., Verschaffel, L., & Onghena, P. (2002). The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. *Journal for Research in Mathematics Education, 33*(5), 319-351.

Van Dormolen, J. (1986). Textuel analysis. In B. Christiansen, A. G. Howson & M. Otte (Eds.), *Perspective on Mathematics Education* (pp. 141-171). Dordrecht: D. Reidel.

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*(2), 122-137.

Wartofsky, M. W. (1979). *Models: Representation and the scientific understanding*. Dordrecht: Reidel, cop.

Weber, R. P. (1990). *Basic content analysis*. Newbury Park, CA: Sage Publications.

Venezky, R. L. (1992). Textbook in school and society. . In P. W. Jackson (Ed.), *Handbook of research on curriculum: A project of the American Educational Research Association* (pp. 437-461). New York: Macmillan Pub. Co.; Toronto: Maxwell Macmillan Canada.

Vergnaud, G. (1987). Conclusion. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 227-232). Hillsdale, NJ: Erlbaum.

Westbury, I. (1990). Textbooks, textbook publishers, and the quality of schooling. In D. L. Elliot & A. Woodward (Eds.), *Textbooks and schooling in the United States: Eighty-ninth yearbook of the National Society for the Study of Education*. Chicago: University of Chicago Press.

Wikipedia. (2010). Factorization. Retrieved November 28, 2010 from http://en.wikipedia.org/wiki/Factorization

Vincent, J., & Stacey, K. (2008). Do mathematics textbooks cultivate shallow teaching? Applying the TIMSS video study criteria to Australian eighth-grade mathematics textbooks. *Mathematics Education Research Journal, 20* (1), 82-107.

Vinogradova, N. (2007). Solving quadratic equations by completing squares. *Mathematics Teaching in the Middle School, 12* (7), 403-405.

Vretblad, A. (2000). *Algebra och Geometri*. Kristianstad: Gleerups.

Zhu, X., & Simon, H. A. (1987). Learning mathematics from examples and by doing. *Cognition and Instruction, 4* (3), 137-166.

Appendix 1: The twelve mathematics textbooks used for the third round of analyses

1. *Matematik 4000 B* (the Blue book) (Alfredsson et al., 2007)
2. *Matematik 4000 B* (the Red book) (Alfredsson, Erixon, & Heikne, 2008b)
3. *Matematik 3000 B* (the Green book) (Björk et al., 2000)
4. *Matematik 3000 B* (the Red book) (Brolin & Heikne, 2003)
5. *Exponent B* (red) (Gennow et al., 2005b)
6. *Exponent B* (yellow) (Gennow, Gustafsson, & Silborn, 2005a)
7. *Matematik från A till E*–gymnasiets matematik kurs B (Holmström, 2001)
8. *Räkna med Vux*–kurs B (Danielsson, Gabrielsson, & Löfstrand, 2002)
9. *Nya delta*–matematik kurs A och B (Björup, Körner, Oscarsson, & Sandhall, 2000)
10. *Δ NT/a+b*–Liber Pyramid. Gymnasiematematik för NV och TP, kurs A och B (Wallin, Lithner, Wiklund, & Jacobsson, 2000).
11. *Origo: matematik kurs B för samhällsvetenskapliga och estetiska program* (Szabo et al., 2008)
12. *Matematik B* (Norberg, Viklund, & Burström, 2002)

Appendix 2: An example of one of the twelve tables containing coded content elements from every investigated textbook

Table 1

Coded content elements in Textbook One

Chapter 1. Algebra and geometry	• Picture and introduction activity • Polynomial conception, coefficient, degrees of polynomials • Polynomial form *p* (*x*) and its value • Commutative property, associative property, distributive property, parenthesis rules • Rules for multiplication of two binomials and signs • Difference of squares rule (konjugatregeln) and perfect square rules or squaring binomials (kvadreringsreglerna) • Factoring polynomials with two methods a) greatest common divisor (no mentioning of the GCD term) b) using difference of square and perfect square rules • Investigating activity-finding models for computing big numbers in squares • Simple quadratic equations and the two solving methods • The method of completing a square • Quadratic formula • Discovering activity-finding relations between roots and coefficients • History • Algebra and application of algebra in real world problems • Geometry sections with Pythagoras' theorem • Discussion activity (right or wrong?) • Homework • Summary of the chapter • Mixed exercises (two) • Problems for everybody
Chapter 2. Functions	• Picture and activity • Definition of function (in forms of value table, graphs or words), • Linear functions with slope formula • Parallel and right angle lines as well as its formula • How to set up a linear equation by a given slope value or values of coordinates • Lab activity • Linear equation of common forms • Linear models expressed by linear functions used for finding the changing factors' influences in the real world problems • Regression-the best linear model • Linear equation system and the three solving methods as well as

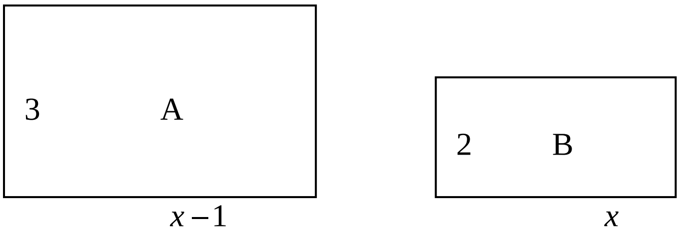
	some special linear equation systems • Investigating activity • Application of linear equation system in real world problem • Inequalities and the solving inequalities • Investigating activity for quadratic equations • Non-linear functions and the characteristics of a quadratic equation: maximum and minimum coordinates, symmetry line, vertex, x-intercepts; finding x-intercepts with algebraic method through solving quadratic equations • Quadratic models-problems in quadratic models solved algebraically and graphically • Non-linear models: a quadratic model and an exponential model • Application of functions in statistics: linear, quadratic, and exponential models • Discussion activities • History • Problems for everybody • Discussion activities (true or false?) • Homework • Summary • Mixed exercises (two)

Appendix 3: The introduction activity on page 7 in the textbook of *Matematik 4000 B* (the Blue book)

Rectangle and algebra:

1. The area of rectangle A can be described with an expression $3(x-1)$.

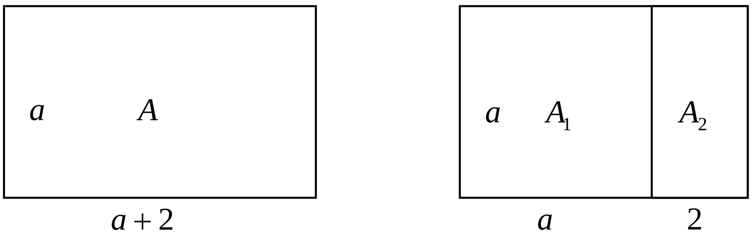
a) Give an expression for the area of rectangle B.



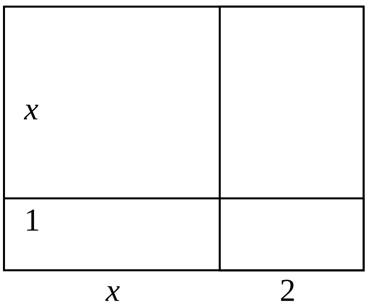
b) Compute areas when x has different value.

c) Have you found any value which gives the rectangles the same area?

2. In the following figure, we get two identical rectangles. Write the equality $A = A_1 + A_2$ with an algebraic expression which corresponds with the respective areas.



3. Write equivalent relationship showing that the area of the whole rectangle is equal to the sum of the four areas for the small rectangles.



4. See the following figure. $(a+b)^2 = ?$