NONCONFORMING ROTATED Q_1 TETRAHEDRAL ELEMENT WITH
EXPLICIT TIME STEPPING FOR ELASTODYNAMICS

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Nonconforming rotated Q_1 tetrahedral element with explicit time-stepping for elastodynamics

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SUMMARY

In this paper we apply a rotated bilinear tetrahedral element recently introduced by Hansbo [1] to elastodynamics in $\mathbb{R}^3$. This element performs superior to the constant strain element in bending and, unlike the conforming linear strain tetrahedron, allows for row-sum lumping of the mass matrix. We study the effect of different choices of approximation (pointwise continuity versus edge average continuity) as well as lumping versus consistent mass in the setting of eigenvibrations. We also use the element in combination with the leapfrog method for time domain computations and make numerical comparisons with the constant strain and linear strain tetrahedra.

KEY WORDS: Finite element method, nonconforming element, explicit time stepping.

1. INTRODUCTION

In [1], we introduced a nonconforming Rotated Bilinear Tetrahedral (RBT) element, derived from the nonconforming hexahedral element introduced by Rannacher and Turek [2]. The RBT element has properties similar to the hexahedral element: improved bending behavior compared to the Constant Strain Tetrahedron (CST), as well as allowing for under-integration in near incompressibility. The shortcomings of the CST element are well known and attempts to improve its properties has typically employed mixed (stabilized) formulations with pressure or stresses as auxiliary variables, as in the Characteristics Based Split (CBS) algorithm of Zienkiewicz et al. [3, 4], or the nodal averaging (NA) approach of Bonet et al. [5, 6, 7], or the sub-grid stabilization method of Cervera et al. [8]. While the CBS approach only addresses near incompressibility, the NA approach additionally gives improved bending behavior. In contrast, the RBT element inherits these properties from its relation with the hexahedral element. No additional variables are introduced, no stabilization necessary, and symmetric solvers such as conjugate gradients can be employed in the stationary case. For explicit solvers, the natural (in the sense that it corresponds to a quadrature method, cf., e.g., Hansbo [9]) mass lumping method based on putting the row-sum in the diagonal of the lumped mass matrix can be used (which is not the case for the conforming linear strain tetrahedron).

An outline of the paper is as follows: in Section 2 we recall the RBT element and discuss two types of continuity imposed on the solution. In Section 3 we give application to the problem of linearized elastodynamics, with mass lumping. Finally, in Section 4 we give some numerical examples to show the properties of the approximation.

2. THE ROTATED Q_1 APPROXIMATION

In order to define a low order approximation which is stable for Stokes problem, Rannacher and Turek [2] constructed a hexahedral element with nodes on the faces. On the reference hexahedron in Fig. 1, we indicate the local numbering of degrees of freedom for this element. Two different kinds of continuity can now be imposed: point-wise continuity and average continuity (we remark that for the classical Crouzeix-Raviart nonconforming tetrahedral element [10], this is in fact the same thing).

On the reference element $\hat{K}$ (for $0 \leq \xi, \eta, \zeta \leq 1$) we define the shape functions by

$$\phi_i = a_0 + a_1\xi + a_2\eta + a_3\zeta + a_4(\xi^2 - \eta^2) + a_5(\xi^2 - \zeta^2).$$

For each shape function, this gives six unknowns a_0 – a_5 for the six faces of a hexahedron element. (It may seem that symmetry requires an additional term $a_6(\zeta^2 - \eta^2)$; note, however, that this term is represented by setting $a_5 = -a_4$).

Enforcing point-wise continuity in the center of the faces, we immediately obtain

$$\begin{aligned}\phi_1 &= \frac{1}{3}(2 + 2\xi - 7\eta + 2\zeta - 2\xi^2 + 4\eta^2 - 2\zeta^2), \\ \phi_2 &= \frac{1}{3}(-1 - \xi + 2\eta + 2\zeta + 4\xi^2 - 2\eta^2 - 2\zeta^2), \\ \phi_3 &= \frac{1}{3}(-1 + 2\xi - \eta + 2\zeta - 2\xi^2 + 4\eta^2 - 2\zeta^2), \\ \phi_4 &= \frac{1}{3}(2 - 7\xi + 2\eta + 2\zeta + 4\xi^2 - 2\eta^2 - 2\zeta^2), \\ \phi_5 &= \frac{1}{3}(2 + 2\xi + 2\eta - 7\zeta - 2\xi^2 - 2\eta^2 + 4\zeta^2), \\ \phi_6 &= \frac{1}{3}(-1 + 2\xi + 2\eta - \zeta - 2\xi^2 - 2\eta^2 + 4\zeta^2).\end{aligned}$$

In [1], we made the observation that there is a reference tetrahedron $\hat{T}$ inscribed in the reference hexahedron $\hat{K}$, shown in Fig. 2. Using a a sub-parametric map $\mathbf{F} : (\xi, \eta, \zeta) \rightarrow (x, y, z)$, based on the shape functions of the P^1 tetrahedron, to a physical tetrahedron T , we map mid side nodes in the reference configuration to mid side nodes in the physical configuration (an isoparametric map gives the same result). This means that the mapping will be *affine* and we have a first approximation scheme in the physical configuration. We can also define a second set of shape functions $\{\psi_i(\mathbf{x})\}$ with edge-wise mean continuity in physical space, defined by setting, on an element T ,

$$\frac{1}{|E_i|} \int_{E_i} \psi_j(\mathbf{x}) ds = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Here E_i is the edge, of length $|E_i|$, associated with local node i , $i = 1, \dots, 6$.

To formalize the finite element method, let $\mathcal{T}_h := \{T\}$ be a conforming, shape regular tetrahedrization of $\Omega \subset \mathbb{R}^3$ and $\mathcal{E}_h$ the set of all element edges in the mesh. We can then introduce non-conforming finite element spaces constructed from the basis previously discussed by defining the two variants

$$V_1^h := \{v : v|_T \circ \mathbf{F} \in Q_1^R(\hat{T}), \forall T \in \mathcal{T}_h; v \text{ is continuous at the midpoint of } E, \forall E \in \mathcal{E}_h\}, \quad (1)$$

with midpoint continuity, and

$$V_2^h := \{v : v|_T \circ \mathbf{F} \in Q_1^R(\hat{T}), \forall T \in \mathcal{T}_h; \int_E v ds \text{ is continuous, } \forall E \in \mathcal{E}_h\}, \quad (2)$$

with mean continuity along the edges.

3. APPLICATION TO ELASTODYNAMICS

3.1. Problem formulation and finite element approximation

We consider the equations of linear elasticity describing the deformation of an elastic body occupying a domain Ω in $\mathbb{R}^3$ during a time interval $(0, t^{\max})$: find the displacement $\mathbf{u}(\mathbf{x}, t) = [u_i(\mathbf{x}, t)]_{i=1}^3$ and the symmetric stress tensor $\boldsymbol{\sigma} = [\sigma_{ij}]_{i,j=1}^3$ such that

$$\left\{ \begin{array}{ll} \boldsymbol{\sigma} &= \lambda \nabla \cdot \mathbf{u} \mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega \times (0, t^{\max}), \\ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \quad \text{in } \Omega \times (0, t^{\max}), \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega_D \times (0, t^{\max}), \\ \mathbf{n} \cdot \boldsymbol{\sigma} &= \mathbf{h} \quad \text{on } \partial\Omega_N \times (0, t^{\max}), \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) &= \mathbf{w}_0(\mathbf{x}) \quad \text{in } \Omega. \end{array} \right. \quad (3)$$

Here $\boldsymbol{\varepsilon}(\mathbf{u}) = [\varepsilon_{ij}(\mathbf{u})]_{i,j=1}^3$ is the strain tensor with components

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

$\nabla \cdot \boldsymbol{\sigma} = [\sum_{j=1}^3 \partial \sigma_{ij} / \partial x_j]_{i=1}^3$, $\mathbf{I} = [\delta_{ij}]_{i,j=1}^3$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, $\mathbf{f}$ and $\mathbf{h}$ are given loads, and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Furthermore, ρ is the density and λ and μ are the Lamé constants. In terms of the modulus of elasticity, E , and Poisson's ratio, ν , we have that $\lambda = E\nu/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$. Finally, $\mathbf{u}_0$ and $\mathbf{w}_0$ are given initial displacements and velocities, respectively.

Introducing the space

$$W := \{\mathbf{v} : \mathbf{v} \in [H^1(\Omega)]^3, \mathbf{v} \text{ is zero on } \partial\Omega_D\},$$

the weak form of (3) can be written: find $\mathbf{u} \in W$ such that

$$\left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \mathbf{v} \right)_\Omega + a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{h}, \mathbf{v})_{\partial\Omega_N} \quad \forall \mathbf{v} \in W, \quad (4)$$

where

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dV, \quad (5)$$

where $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{ij} \sigma_{ij} \tau_{ij}$ denotes tensorial contraction, and $(\cdot, \cdot)_S$ denotes the L_2 scalar product over S .

To define the finite element method, we introduce the non-conforming finite element space constructed from the spaces V_i^h in (1)–(2) by defining

$$W_1^h := \{\mathbf{v} : \mathbf{v} \in [V_1^h]^3, \mathbf{v} \text{ is zero at the midpoints of edges on } \partial\Omega_D\},$$

and

$$W_2^h := \{\mathbf{v} : \mathbf{v} \in [V_2^h]^3, \int_E \mathbf{v} ds \text{ is zero at the edges on } \partial\Omega_D\}.$$

The (semi-discrete) finite element method is to find $\mathbf{u}_h \in W_i^h$ such that

$$\left(\rho \frac{\partial^2 \mathbf{u}_h}{\partial t^2}, \mathbf{v} \right)_\Omega + a_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{h}, \mathbf{v})_{\partial\Omega_N} \quad \forall \mathbf{v} \in W_i^h, \quad i = 1 \text{ or } 2. \quad (6)$$

Here

$$a_h(\mathbf{u}, \mathbf{v}) := \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dV \quad (7)$$

has to be written as a sum over the elements due to the nonconformity of the approximation. The convergence and stability of this method follows from the analysis in [1].

We shall also consider the discrete time-harmonic eigenvalue problem of finding $\mathbf{u}_h \in W_i^h$ and $\omega \in \mathbb{R}^+$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) - \omega^2 (\rho \mathbf{u}_h, \mathbf{v})_\Omega = 0 \quad \forall \mathbf{v} \in W_i^h, \quad i = 1 \text{ or } 2. \quad (8)$$

This eigenvalue problem is of interest in itself, but will here only be used in order to ascertain the apparent stiffness of different methods.

3.2. Mass lumping

For the time dependent problem, we end up with a matrix system of the type

$$\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{S} \mathbf{u} = \mathbf{f}, \quad (9)$$

and for the eigenvalue problem

$$(\mathbf{S} - \omega^2 \mathbf{M}) \mathbf{u} = \mathbf{0},$$

where $\mathbf{S}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix. For explicit time stepping schemes, it is important that the mass matrix can be lumped to a diagonal matrix $\mathbf{M}^L$. The basic idea of mass lumping is to use an integration scheme approximating the integrand with the shape functions in use (see [9]). Thus, for V_1^h the integral over T of a given function $f(\mathbf{x})$ is approximated

$$\int_T f(\mathbf{x}) dV \approx \sum_{i=1}^6 \int_T \varphi_i(\mathbf{x}) f(\mathbf{x}_i) dV = \sum_{i=1}^6 W_i f(\mathbf{x}_i),$$

where

$$W_i := \int_T \varphi_i(\mathbf{x}) dV.$$

Replacing $f(\mathbf{x})$ by the integrands of the mass matrix, $\varphi_j(\mathbf{x})\varphi_k(\mathbf{x})$, we see by the properties of the shape functions that we are left with a diagonal matrix containing the integration weights W_i in

the diagonal. The integral of a shape function can also be found by summing the corresponding row in the full mass matrix (row-sum lumping), since

$$\sum_j \int_{\Omega} \varphi_i \varphi_j dV = \int_{\Omega} \varphi_i \sum_j \varphi_j dV$$

and $\sum_j \varphi_j = 1$. The weights W_i are thus immediately obtained given the full mass matrix. For V_2^h the situation is slightly more complicated. If we define the mean value of $f(\mathbf{x})$ along edge E_i by

$$\pi_{0,i} f := \frac{1}{|E_i|} \int_{E_i} f(\mathbf{x}) ds$$

we can make the approximation $f(\mathbf{x}) \approx \sum_{i=1}^6 \psi_i(\mathbf{x}) \pi_{0,i} f$, which we note is exact if $f \in V_2^h$, and

$$\int_T f(\mathbf{x}) dV \approx \sum_{i=1}^6 \int_T \psi_i(\mathbf{x}) \pi_{0,i} f dV.$$

Setting now $f = \psi_j \psi_k$, we have

$$\int_T \psi_j \psi_k dV \approx \sum_{i=1}^6 \left(\int_T \psi_i dV \frac{1}{|E_i|} \int_{E_i} \psi_j \psi_k ds \right),$$

which does not yield a diagonal matrix. We thus make an additional assumption:

$$\int_{E_i} \psi_j \psi_k ds \approx \int_{E_i} \psi_j \pi_{0,i} \psi_k ds = |E_i| \pi_{0,i} \psi_j \pi_{0,i} \psi_k = \begin{cases} |E_i| & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We note that this would be exact if ψ were linear (cf. the Crouzeix-Raviart element in 2D [10]). The resulting lumped mass matrix thus also has entries of the type

$$\tilde{W}_i = \int_T \psi_i(\mathbf{x}) dV$$

and these can be taken from the row sum of the corresponding full mass matrix, since $1 = \sum_i \psi_i \pi_{0,i} 1 = \sum_i \psi_i$.

The entries in the diagonal of the lumped element mass matrices are given explicitly by $W_i = \tilde{W}_i = |T|/6$, where $|T|$ denotes the volume of T . Since one point integration per element, using derivative information, cf. [11], can be used for evaluating the stiffness matrix (and, since the mapping is affine, is indeed exact assuming constant material data), the implementation of the RQT element can be made competitive with the CST element.

Remark 3.1. *It may seem like an unnecessary complication to introduce mean edge continuity because it seems nothing is gained in comparison with point-wise continuity. One benefit lies in that, just like a discrete version of Gauss' theorem is gained for face continuity in the Rannacher-Turek element, cf. [2], we retrieve a discrete version of Stokes' theorem with edge continuity. That is, for a given displacement field $\mathbf{u}$ we have an edge interpolant $\pi_0 \mathbf{u}$ (of edge wise mean values of $\mathbf{u}$) such that for a face F on the tetrahedron,*

$$\int_F \mathbf{n}_F \cdot (\nabla \times \mathbf{u}) dA = \oint_{\partial F} \mathbf{u} \cdot d\mathbf{r} = \oint_{\partial F} \pi_0 \mathbf{u} \cdot d\mathbf{r} = \int_F \mathbf{n}_F \cdot (\nabla \times \pi_0 \mathbf{u}) dA,$$

where $\mathbf{n}_F$ is a unit normal to F and $d\mathbf{r}$ follows the right-hand rule. Thus the edge oriented method has additional structure (giving bending resistance) built into it, and can be expected to be stiffer than the point oriented method (which, experiments show, tends to underestimate the actual stiffness). This will be further commented in the numerical examples below.

3.3. Time stepping

For the explicit time stepping, we use the “position Verlet” version of the leapfrog method applied to (9), i.e., for equidistant time stepping with step size $k = t_{n+1} - t_n$, and with $\mathbf{u}_n \approx \mathbf{u}(t_n)$ and $\mathbf{v}_n \approx d\mathbf{u}(t_n)/dt$, we solve, for $n = 0, 1, \dots, n_{\max}$ (where $t^{\max} = k n_{\max}$),

$$\begin{aligned}\mathbf{u}_{n+1/2} &= \mathbf{u}_n + \frac{k}{2} \mathbf{v}_n, \\ \mathbf{M}^L \mathbf{v}_{n+1} &= \mathbf{M}^L \mathbf{v}_n + k (\mathbf{f}(t_{n+1/2}) - \mathbf{S} \mathbf{u}_{n+1/2}), \\ \mathbf{u}_{n+1} &= \mathbf{u}_{n+1/2} + \frac{k}{2} \mathbf{v}_{n+1}.\end{aligned}$$

As is well known, cf., e.g., [12], this requires a time step size $k \leq 2/\omega_{\max}$, where $\omega_{\max}$ is computed from the eigenvalue problem $(\mathbf{S} - \omega^2 \mathbf{M}^L) \mathbf{u} = \mathbf{0}$.

4. NUMERICAL EXAMPLES

4.1. Eigenfrequencies

We consider a problem analyzed in [13], consisting of a hexahedron with side lengths L_x , L_y , and L_z along the corresponding axes, with slip boundary conditions on each side (i.e., with the normal displacement prescribed to zero on each face of the cube). Assuming constant material data, the eigenvalues are then given analytically by

$$\omega^2 = \xi \frac{E}{2\rho(1+\nu)(1-2\nu)}$$

where, with

$$\alpha_n = \frac{n\pi}{L_x}, \beta_n = \frac{n\pi}{L_y}, \gamma_n = \frac{n\pi}{L_z}, n = 0, 1, \dots,$$

we obtain a set of eigenvalues associated with solenoidal modes:

$$\xi_{ijk}^s = (1-2\nu)(\alpha_i^2 + \beta_j^2 + \gamma_k^2),$$

and another set of eigenvalues associated with compressive modes:

$$\xi_{ijk}^c = 2(1-\nu)(\alpha_i^2 + \beta_j^2 + \gamma_k^2), \text{ with } i+j+k \neq 0.$$

The multiplicity of the eigenvalues is also given in [13]. We set $\rho = 7850 \text{ kg/m}^3$, $E = 200 \text{ GPa}$, and $\nu = 0.33$, and compare the convergence of the lowest eigenvalue for the point oriented, edge oriented, and CST element, with and without mass lumping, in Fig. 4. The point oriented approximation gives the smallest error, and the CST element is the stiffest. Note that we are measuring the error with respect to degrees of freedom (NDOF). In Fig. 5 we give the quotient of the exact eigenvalue to the approximate, and we note that the point-oriented method (with lumping) may underestimate the eigenvalue. This is unlike a standard conforming FEM, where the eigenvalues are guaranteed to be over-estimated.

4.2. Explicit time stepping

We consider free vibration in different modes: compressive, bending, and twisting. The domain is a console of dimension $(0, 1) \times (0, 0.1) \times (0, 0.1)$, fixed at $x = 0$. The compression mode is obtained by releasing the displacements corresponding to a volume load $\mathbf{f} = (-10^{11}, 0, 0)$, Fig. 6, the bending mode corresponds to the lowest eigenvector, Fig. 8, and the twisting mode is obtained by releasing the displacements corresponding to $\mathbf{f} = (0, -10^{11}(z - 0.05), 10^{11}(y - 0.05))$, Fig. 10. The deformation plots also give information about the mesh used.

The different initial conditions are computed using the linear strain tetrahedron, and the solution is transferred to the nodes of the linear polynomials and the rotated Q_1 polynomials for comparison. Thus similar starting conditions is used for each of the three elements. We compare time plots for the different cases using the same time step (the critical time step computed for the linear strain tetrahedron) over a few periods. The material data are the same as previously.

Note that the linear strain tetrahedron cannot be lumped and thus does not allow for a fully explicit scheme; we use $\mathbf{M}$ in the place of $\mathbf{M}^L$ for this element.

In Fig. 7 we see that for pure compression, all approximations give essentially the same result. For bending, Fig. 9, the constant strain tetrahedron exhibits a marked period shortening due to its spurious bending stiffness. The edge oriented approximation is slightly stiffer, and the point value oriented is slightly more flexible, than the linear strain tetrahedron, as expected. The situation is the same, only more pronounced, in the twisting case, Fig 11.

5. CONCLUDING REMARKS

The rotated Q_1 tetrahedron element allows for two different approximations: one point value oriented, with continuity in the midpoint of the edges, and one edge oriented, with mean values along the edges as unknowns. Both approximations allow for mass lumping. Numerically we observe a slightly overly flexible response, compared to the linear strain element, for the point-oriented method, and a slightly overly stiff response for the edge-oriented. Both approximations perform distinctly better than the constant strain element in bending dominated situations.

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Figure 1. Reference hexahedron $\hat{K}$.

Figure 2. Reference element $\hat{T}$ inscribed in the hexahedron of Fig. 1.

Figure 3. Physical element T .

Figure 4. Convergence of the first eigenmode.

Figure 5. Quotient between exact and approximative first eigenvalue.

Figure 6. Compression mode.

Figure 7. Response for compression mode.

Figure 8. Bending mode.

Figure 9. Response for bending mode.

Figure 10. Twisting mode.

Figure 11. Response for twisting mode.