How Irrational Behaviour Creates Order and How This Order Can Be Determined
The Theory and Practice of Fractal Market Analysis

Bachelor’s Thesis within Economics
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Abstract

This paper analyzes two main frameworks that challenge the “mainstream” finance theory and the random walk hypothesis. The first framework is based on investor irrationality and is called Behavioural Finance. The second framework views the financial market as a chaotic system and is called Fractal Theory of a financial market.

Behavioural Finance attacks the assumption of investor rationality, thus challenging the conventional finance theories on the micro level. Fractal Theory challenges the EMH and the “macroeconomics” of finance. This paper presents a step towards unifying the frameworks of Behavioural Finance and Fractal Theory.

After a review of the relevant literature, a model of the financial market is suggested that rests on the predictions of both Behavioural Finance and Fractal Theory. As a next step, a mathematical algorithm is described that allows to test the financial market for consistency with the presented model.

The mathematical algorithm is applied to 10 years of daily S&P500 price quotes, and consistent statistical evidence shows that the predicted fractal pattern reveals itself in the S&P500 prices. The new model outperforms the random walk in out-of-sample forecasting.
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1 Introduction

Throughout modern economic history, financial markets have been assumed to follow a random walk process. Since its mathematical formalization in 1900, the random walk has been accepted as the “best” model of the financial market, and financial theories are still being formed based on this assumption. Arguably the most prominent economic conceptualization of the random walk is the Market Efficiency Hypothesis, or the EMH. Yet another historically accepted view in economics, complementary to the EMH, is that of investor rationality (and utility-maximization), which has eventually lead to the introduction of the Capital Asset Pricing Model, or CAPM. During further theory build-up, the prices of complex derivative instruments were calculated, introducing derivative markets into the financial system, all based primarily on the two assumptions above.

In the course of the past thirty years, the persistent recurrence of price bubbles and financial crises compiled persuasive evidence against this “mainstream” financial framework and its underlying assumptions. As a result, alternative models and theories have been developed that might have the potential to challenge the entire contemporary financial system. So far the popular theories of financial economics were challenged separately on the micro and macro level. On the micro level, the challenging of the assumption of investor rationality has lead to the emergence of Behavioural Finance. On the macro level, evidence against the random walk in market prices has been conceptualized and analyzed through Fractal Theory.

Behavioural Finance creates an image of the stock price movement as a heterogenous process with price oscillations, momentum and bubbles, all of which result from an evolutionary game between (quite irrational) agents. Fractal Theory tackles the market price movements as a chaotic process with fractal structure, i.e. a locally unpredictable process that resembles itself on multiple scales of time and magnitude.

This paper attempts to make a step towards reconciling and unifying the theories of Behavioural Finance and Fractal Theory into one paradigm in the same way as the assumption of investor rationality and the random walk model have been unified into the EMH, and later into CAPM. A new (fractal) model of a financial market is thus developed that incorporates the predictions under both Behavioural Finance and Fractal Theory.

In order to test the predictions under the new fractal model, a new Fourier transform-based algorithm for harmonic market analysis is introduced that can be applied to non-stationary time series data. The new algorithm is applied to 10 years of daily quotes of the S&P500 market index. While the main purpose of this paper is a comparative discussion of finance theory, potential further applications include portfolio allocation and derivative pricing, as well as the prediction and prevention of price bubbles on a large scale.

I proceed by discussing the theoretical advancements before and within the area of Behavioural Finance. After that, I discuss the ways in which processes consistent with the Behavioural Finance framework can translate into fractal-like chaotic patterns on the macro level, thus unifying the predictions of Behavioural Finance with the findings of Fractal Theory. The subsequent sections present the derivation and discussion of the technical model, proceeding with the results of its application to the S&P500 index. The concluding section discusses the results and gives suggestions for further research.
2 Financial Market Theory

This section proceeds by briefly summarizing the advancements of finance theory that have lead to the introduction of the EMH and the emergence of its rival theories. After that, the relevant theories and models from Behavioural Finance are considered in more detail. The subsequent sections attempt to establish the transition from the microscopic game of Behavioural Finance to the macroscopic fractal structure of a financial market, thus unifying the conclusions from Behavioural Finance and Fractal Theory into one testable model of a financial market system.

2.1 History

Starting with the famous work of the French mathematician Louis Bachelier ‘Théorie de la Speculation’ (1900), the movement of stock market prices has been commonly viewed as a largely random stochastic process. As summarized by Granger and Morgenstern (1963), 'The various authors, though starting from different viewpoints and developing slightly different hypotheses, have been surprisingly consistent in their general conclusions', namely, that markets follow a random walk expressed as $P_{t+1} = P_t + \varepsilon_{t+1}$, where $P_t$ denotes the observation of a price series at time $t$, and $\varepsilon$ is an i.i.d. error term $\sim N(0, \sigma)$, commonly referred to as white noise.

Following this mathematical model, numerous respectable economists, most distinguishably Eugene F. Fama (e.g. 1965, 1970 and 1998), have argued that financial markets are generally efficient in responding to all available information, thus developing and defending what is now referred to as the Efficient Markets Hypothesis (EMH) [Fama (1970)]. The EMH viewpoint implies that speculation is generally pointless, and technical analysis has no forecasting capabilities.

In the past two decades, the fundamental theories of finance have taken a hit of criticism. The assumptions underlying the Efficient Market Hypothesis (EMH), the Capital Asset Pricing Model (CAPM) [Sharpe (1964); Lintner (1965); Black (1972)], Markowitz’s Portfolio Theory [Markowitz (1959)] and the Black-Scholes Option Pricing Formula [Black & Scholes (1973)] were called into question in much academic literature. The criticism of the EMH and CAPM is generally based on questioning the micro-economic assumption of investor rationality, the cornerstone of traditional economics. The set of alternative economic theories and models that are based on investor irrationality and systematic psychological biases is commonly referred to as Behavioural Finance.

Behavioural Finance concludes that stock markets exhibit systematic deviations from the “true” (mathematical) expectations, revealed in the form of market over- and under-reactions to news, positive feedback in stock returns, and other processes that translate into volatility increases [e.g. Daniel et al. (1998)]. Behavioural Finance also explains price bubbles and financial crises through herding and information cascades (to be discussed under ‘Recent Advancements’).

Additional systematic criticism of the assumptions underlying the “mainstream” finance theory can be found throughout the academic work of Benoit Mandelbrot, the author of the Fractal Theory in geometry who also applied his analysis to the financial markets, with a degree of success. It is even argued in some literature that fractal theory and the EMH are equally robust models of a financial market [e.g. Yalamova & McKelvey (2009)]. Mandelbrot’s research concludes that markets tend to behave in similar pat-

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1 There are also more technical considerations to why the validity of modern finance theory is questionable. For example, the criticism of Markowitz’s Portfolio Theory and the Black-Scholes formula sometimes refers to their oversimplification of the statistical properties of such random financial variables as stock prices and returns. To give an example, Laloux et al. (1999) argue that Markowitz’s method of calculating the covariance matrix from a vector of returns misstates the systematic risk by treating the returns observations as constants, whereas in reality each return has a statistical distribution. There are also alternative valuation methods for derivative instruments which model prices using ARCH distributions.
terns on different time scales. In other words, the patterns exhibited by market prices on a daily, monthly, yearly etc. basis are similar in certain mathematical respects. Fractal Theory argues that market volatility is understated by the "Random Walk" model, or any prediction based a Gaussian (white noise) distribution of the stock returns.

### 2.2 Recent Advancements in Behavioural Finance

When making financial decisions, our profit expectations are inevitably based on a tradeoff between risk and return [e.g. Fama (1970)]. Misspecifying risk will lead us to require suboptimal profits. The EMH essentially suggests a random walk (or a geometric random walk) as the ‘correct’ model of stock market behaviour, implying a lognormal distribution of the stock prices (equivalent to a normal distribution of stock returns) and dismissing any deviations from this model as minor and accidental. Portfolio theory, CAPM and the Black-Scholes option pricing formula follow the EMH in dismissing the deviations of risk from the values implied by a lognormal price distribution, in addition to introducing a number of additional strong assumptions, e.g. return observations are perfectly accurate (Markowitz), markets do not gap (Black-Scholes), all agents are rational and hold efficient portfolios (CAPM), etc.

Contemporary financial market theory is based on generating sophisticated models of market behaviour where the unrealistic assumptions underlying the EMH, CAPM and the Black-Scholes formula start to be relaxed. One can speculate that the least realistic (and the most conveniently simplifying) assumption is that of investor rationality. When the model of a rational utility-maximizing investor is dismissed, the picture of the financial market becomes infinitely more complex. One of the rewards of considering the financial markets under such complexity is that predictable patterns reveal themselves in irrational human behaviour.

#### 2.2.1 Investor Sentiment and Market Overreactions

Investigations of systematic irrational behaviour can be traced as far back as 1841, when the book ‘Extraordinary Popular Delusions and the Madness of Crowds’ by the Scottish journalist Charles MacKay was first published. Among other “popular delusions”, MacKay considers the Tulipmania in early XVII-century Holland, which is now the earliest well-documented asset bubble in financial history (to be discussed later).

Despite an early discovery, the psychological considerations in finance have not received much attention until the work of Daniel Kahneman and Amos Tversky (1973, 1974, 1979 etc.), describing a number of heuristics and biases that are common to human decision-making. The work by Kahneman and Tversky has resulted in what is now referred to as Prospect Theory and the value function, inspiring a number of financial market models that involved a game between rational investors (also referred to as arbitrageurs or fundamentalists) and irrational investors (also referred to as trend traders, noise traders etc.).

To give a few examples, De Long et al. (1990a, 1990b) present a simple mathematical model that assumes that noise traders misperceive the variance and expected return of a security, whereas the rational investors make accurate predictions. Both groups make investment decisions accordingly and take prices as given. There is only one security and a risk-free asset. Factoring in consumption and introducing two overlapping generations, De Long et al. (1990a) show that the group of irrational traders outperforms the group of rational traders by (accidentally) taking on more risk, despite the fact that the irrational traders hold inefficient portfolios, while the rational traders make optimal investment decisions.

The above model assumes that noise traders take prices as given, which means that it does not shed light on the systematic deviations of the financial markets from a random walk process. This flaw is corrected in De Long et al. (1990b), where noise traders are allowed to affect prices. The new model shows that in the world where the actions of noise traders affect the volatility of stock prices, noise traders still domi-
nate the market as long as their erroneous expectations are biased upwards. Each subsequent generation, the noise traders gain in their share of the market, and as a result the volatility increases even further.

What could be perceived as the next step in noise trader modeling is momentum in irrational thinking. A prominent model of accumulated investor sentiment can be found in Barberis et al. (1998). This model is based, among other theories, on the representativeness heuristic suggested by Kahneman and Tversky, and specifically what is later called the misconception of regression [Kahneman & Tversky (2000)]. Barberis et al. model noise trader behaviour by a propensity to underreact to one piece of information such as good or bad news, while overreacting to a series of good or bad news announcements.

Statistically, as news is generally believed to be normally distributed in time, some (positive or negative) news events sometimes appear in clusters. The misconception of regression forces a person to perceive a recent (accidental) cluster of random occurrences as evidence of a trend. In financial terms, a noise trader may perceive some portion of a perfect random walk (in e.g. stock prices) as an upward or downward trend and make investment decisions accordingly. This type of behaviour is commonly referred to as trend trading. The model by Barberis et al. leads to an intuitive conclusion that when a large enough number of traders engage in trend trading their actions become a self-fulfilling prophecy. As a result, the financial market will exhibit the same systematic bias that the traders in that market are subject to.

Another study performed in the same year presents additional evidence that market overreactions to news exhibit momentum. Daniel et al. (1998) suggest a model that factors in overconfidence and self-attribution bias in traders. The model’s central assumption is not of one group of traders being irrational, but of one group of traders receiving “noisy” private signals. The ill-informed traders are overconfident about the received information. Additionally, these investors may be overconfident about their trading skills (i.e. exhibit self-attribution bias), which will exacerbate the market overreactions even further. Figure 1 below shows the suggested response of the market to new information, both under rational expectations and when the investors exhibit overconfidence and self-attribution bias.

![Figure 1: Market reaction to news under investor overconfidence; without (solid line) and with (dashed line) self-attribution bias in ill-informed investors.](source: Daniel et al. (1998))

The results of the Barberis et al. model and the Daniel et al. model jointly suggest that both noise traders and professional traders are subject to systematic psychological biases, leading to determinstic patterns in market behaviour. Both the rational and irrational groups suffer from a biased perception of the intrinsic value of the (random) new information about the market. As a result, the market also responds to new information in persistently inefficient ways.
2.2.2 Positive Feedback\(^2\) and Herding

Although positive feedback is a generic term for describing processes that do not self-regulate or revert to their mean, the term has been applied (interchangeably with price-to-price feedback, or simply feedback) to the Behavioural Finance theory of price bubbles. The idea behind (positive or price-to-price) feedback theory is simple, as it incorporates solely the fact that positive public information under efficient information exchange (such as word-of-mouth, for example) factors into public expectations about future prices. This results in a simple multiple generations model where the first investor generation’s success increases the attention (and positive expectation) of the next investor generation. The next generation repeats the first generation’s investment strategy and increases the price of the portfolio (or single security) by increasing speculative demand. If this feedback process is not interrupted over a large enough number of generations, an asset bubble is created.

The first well-documented case of an asset bubble is the already mentioned Tulipmania in medieval Holland, when the price of a tulip increased approximately twenty times and then fell back down to its initial level, all within a period of six months. It can be argued that among the later cases of positive feedback are the dotcom bubble of 2000-2001 and the subprime mortgage bubble of 2007-2008.

Another way to look at the price bubble phenomenon is through clusters of investor activity, or herding [e.g. Welch (1992)]. While there are several identified types of herding activity, the basic unifying idea is that it is common, and at times even rational, for investors and other economic agents to mimic noise traders or even make suboptimal decisions\(^3\). For example, if a rational investor knows that an asset bubble is being formed on the basis of irrational expectations, it is still optimal to bet on bubble growth and withdraw the money (or even go short) just before the bubble burst, rather than betting on a price correction at the time of the bubble’s birth.

2.2.3 Generalizing and Simulating Irrational Behaviour

As viewed through the loop of Behavioural Finance, financial markets are essentially complex systems of transactions involving interactions between agents and expectations. From this perspective, the entire financial market can be broken down into individual transactions made on the basis of a game between (rational and irrational) players with different strategies and time horizons. Conversely, an entire financial market can be modeled by accurately modeling a group of players and forcing them to interact. The models of irrational and suboptimal investor behaviour described above, along with a number of other models, provide a framework according to which systematic psychological biases can be introduced into computer simulations. In this way, a fairly accurate artificial model of a financial market can be derived.

Steinlitz & Shapiro (1996) (S&S) perform a microsimulation of a trading game with two commodities, food and gold. There are three categories of agents: regular agents produce and consume only (thus forming the fundamental value of the two goods); value traders trade based on their perception of the fundamental values of the commodities; trend traders trade based on their perceptions of a trend in the price levels. S&S show that when the (exogenous) fundamentals are forced into random movement, the market forms price bubbles on the upward movements in the fundamentals, while forming deflationary bubbles on the downward movements of the fundamentals.


\(^3\) A reference can be made here to the mathematical concept of the Nash equilibrium (and in particular the Cournot-Nash equilibrium) that mathematically proves that the optimal decision for a group is, under certain conditions, suboptimal for a single member of the group.
A later simulation performed by Stegitz and O’Callaghan (1997) introduces a proxy for economic growth by introducing additional gold each period into the system described above, thus making the trend traders more active. The results of the new simulation show that with an exogenous increase in the wealth of trend traders, price bubbles form independently of the movements in the fundamentals. In the simulation, asset bubbles (positive) are formed and burst, followed by a sharp price correction, a subsequent increase in volatility, and eventually a deflationary bubble (negative), after which the system comes back to normal (see Figure 2 below).

![Figure 2: A modeled asset bubble followed by a deflationary bubble](source: Stegitz and O’Callaghan (1997))

A yet more complex simulation of the financial market was introduced by Palmer et al. (1994) that factored in a proxy for evolution and learning of the agents in the system. This new model was eventually conceptualized and given the name Minority Game. The initial Minority Game model inspired numerous researchers to generate their own Minority Game-based simulations, among which the model by Giardina and Bouchaud (2003) can be distinguished.

Giardina and Bouchaud endow each agent with a choice of strategies, among which is also a choice of not participating in the market. Each agent chooses between strategies heuristically (i.e. based on the past effect of these strategies on the agent’s wealth). The model introduces arbitrary values for such factors as the time intervals used by traders for trend recognition, the propensity for irrational behaviour, the time taken to trigger fundamental trading, the level of information entropy etc. By finally factoring in the extent to which traders affect market prices and the extent to which traders prefer to follow trends (trend trading), Giardina and Bouchaud discover three alternative market phases for different arbitrary parameter values: the oscillatory (or periodic) phase, the stable phase and the intermittent phase (see Figure 3 below). The intermittent phase resembles the “real” stock market by several parameters, including the presence of volatility clusters, the distribution of the returns and the heterogeneity of the stock price behaviour.

![Figure 3: Price charts for three stock market regimes](source: Giardina & Bouchaud (2003))

The behaviour of the modeled stock prices against time under purely random strategies is as follows:
The game-based simulations of a stock market shed light on the emergence of asset bubbles, deflationary bubbles and price oscillations. According to the results of these simulations, under a set of assumptions from Behavioural Finance the self-fulfilling prophecies of irrational investors are statistically likely to lead to these investors outperforming the rational traders as a group, which will result in an increase in the share of irrationality in the stock market in the long run. However, as irrational investors learn, the system stabilizes and becomes sustainable. The resulting equilibrium is far from the assumptions of the EMH, CAPM and other conventional finance theories.

2.3 Fractal Theory and the Macro Level

Despite its general mathematical complexity, one can describe the Minority Game simulation in relatively simple terms. While the behaviour of stock prices is determined by the joint behaviour of the agents in the market, the behaviour of those agents is determined by their ability to learn by induction. At the same time, the agents’ inductive learning itself is determined by a history of the stock prices they had previously collectively determined$^4$. This relationship creates a complex and dynamic feedback system, where the very nature of the dynamic feedback is determined by two factors: the initial conditions of the system, and a (finite) number of fixed parameters that “tweak” the agents’ inductive learning. In other words, the movement of the stock prices in the Minority Game simulation is by definition a chaotic process$^5$.

A chaotic process arises when the movements in a variable is interconnected with the movements of other variables through a complex mathematical relationship. A minor change in the nature of the relationship or the starting point of the variable’s movement can potentially lead to vast differences in the nature of the chaotic process. The problem with chaotic movement is that it is largely impossible to predict despite its essentially deterministic nature. Hence, the results obtained in the irrationality-based market simulations also can not be readily translated into powerful mathematical models that would rival such frameworks as CAPM or Portfolio Theory. In simple terms, while being more realistic and providing more insight into the behaviour of the financial markets, the game-based simulations also provide nearly unpredictable outputs that do not have any statistical correlation with the real data. Consequently, the main anomaly that undermines the traditional financial economics framework, namely, price bubbles, al-

$^4$ In a mathematical vinacular, this system of stock price determination can be described as a path-dependent process.

$^5$ For an introduction into the concept of a chaotic process, see Savit (1988).
though better analyzed through Behavioural Finance, remains nearly as unpredictable under this new approach.

Where Behavioural Finance fails to outperform traditional theory, discoveries under the Fractal Theory framework are a potential remedy. The chaotic nature of the stock price movements that is implicitly assumed in Behavioural Finance suggests a connection between Behavioural Finance and Fractal Theory of a similar form as can be observed between investor rationality and the EMH. While Behavioural Finance establishes a chaotic structure of the stock prices on the micro level, Fractal Theory investigates the chaos of the market prices on the macro level.

2.3.1 The Macroeconomics of Human Behaviour

Fractal chaos theory has an elegant solution reconciling the unpredictability of a single random occurrence (such as in this case an asset bubble or a deflationary bubble) in a complex chaotic system. In his book called 'Fractal market analysis: applying chaos theory to investment and economics', Edgar E. Peters provides an intuitive rationale behind this solution, explaining the deterministic approach to random phenomena as being unable to predict when and where a single leaf grows in a tree, but being able to infer what the tree will look like as a whole. This transition from 'local randomness' to 'global structure' (page 5) is what could arguably be observed in the financial market, which is essentially a vastly complex system of chaotic games between traders with different strategies and time horizons.

As long as one accepts the assumption of a fractal nature of the financial markets, predicting such event as a price bubble in a composite financial instrument such as a market index becomes a possible task. Due to the deterministic patterns exhibited in chaos on the macro level, a system arising from an interaction between a large enough number of similar chaotic processes can theoretically be described in a relatively accurate way despite the unpredictability of the individual chaotic components. Moreover, an accurate description of the macrostructure of market chaos may even allow for relatively accurate out-of-sample predictions.

It can be speculated that this point on the line of argument is the crisis of the modern finance theory. Quite frankly, at this juncture the random walk model would probably be the first suggestion to come to mind as to the chaotic financial market’s macroscopic structure that reveals itself through fractal macrostructure. However, in the context where price movements exhibits such complexity on the micro level, the random walk model is by no means the true model, as it assumes identical simplicity in stock price behaviour on all scales, prohibiting chaos and irrationality altogether.

Hence, for the purpose of a further advancement in market theory, it is necessary to first assume that the fractal structure that reveals itself in the chaos of market prices only resembles a random walk process. Moreover, it may need to be suggested that this fractal structure only resembles randomness as such, being in actuality a complex path-dependent (chaotic) process with a complex and yet discernable deterministic macrostructure. From this point on, being able to determine the relationships through which this macrostructure arises would theoretically imply being able to formulate this macrostructure in a testable way.

2.3.2 A Portfolio of Assets with Chaotic Prices

On the micro level, i.e. the level of one security and a transaction-by-transaction process, the chaotic game of price determination translates into a seemingly random discrete probability distribution of price bubbles in time (this can also be seen in Figures 2 and 3: Intermittent Stage). This discrete distribution of price bubbles in a single security does not allow for an accurate prediction of the next bubble event.  

\[ It\ \text{is} \ \text{indeed} \ \text{difficult} \ \text{at} \ \text{times} \ \text{to} \ \text{draw} \ \text{a} \ \text{distinction} \ \text{between} \ \text{a} \ \text{chaotic} \ \text{process} \ \text{and} \ \text{a} \ \text{stochastic} \ \text{one} \ (\text{that is, given the assumption that such a distinction is not a subjective one to begin with). For a discussion on how a chaotic process can be distinguished from a completely random process, see Savit (1988).} \]

8
However, the statistical properties of this bubble distribution is bound to change when a large number of assets is combined into one portfolio. In order to make an illustrative example, let us consider an equal-weighted portfolio containing $n$ assets with chaotic prices (i.e. assets like that that described in the Minority Game). Such an asset portfolio will contain all of the price bubbles formed in each security, which means that the density of the asset bubbles will dramatically increase, while their effects will decrease proportionally to the portfolio size.

If price bubbles were sampled jointly from all assets in the portfolio, their distribution would be expected to approach normality as $n$ increases. However, within the portfolio the bubbles will interact, and their effects will add up. More precisely, in the case where two or more bubbles overlap in time, their effects are bound to either cancel out (if one is an asset bubble and the other is a deflationary bubble), or create resonance (if both are asset bubbles or deflationary bubbles). When the bubbles cancel out, the aggregate price of the portfolio will remain relatively steady. When the bubbles resonate, the resulting spike will be higher.

A resonance between a number of small price bubbles would in turn have the potential to form another price bubble, only of a much larger magnitude. Moreover, one can suggest that the occurrence of a larger price bubble in a resonance of smaller bubbles in the portfolio will follow the same (discrete) probability distribution as the initial small bubbles in the a single security, only on a much larger scale.

2.3.3 From Heterogenous Investment Horizons to Scalability and Fractals

The system described above essentially represents a transition from chaos (individual security prices), to discrete random events (individual security price bubbles), to a normal distribution, and back to (a larger and more complex scale of) chaos and discrete random events. Now let us consider a model where the (chaotic) prices of individual securities are determined by millions of agents with heterogenous investment horizons and resources available. Each class of investors will enter the Minority Game on their own time scale and will use their own amount of available resources to derive an optimal investment strategy under imperfect information entropy.

Presumably, what this heterogenous system will create is in essence a pyramid of transitions between chaos and normality on multiple scales. These pyramids will interact to potentially form new levels of price bubbles, and a large enough number of price bubbles will result in a normal distribution on yet another scale. What would then essentially emerge is a fractal structure.

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7 This follows from basic portfolio theory. In an equal-weighted portfolio with "n" assets, the weight of each asset is precisely "1/n", which implies that the effect of each asset's price movements on the price of the portfolio will be exactly "1/n" the effect of the same asset's price movements on the price of that asset.

8 In more mathematical terms, this assumption follows from the central limit theorem. We can infer that in a single security (e.g. a company stock), price bubbles of every given magnitude range follow a Poisson distribution under a fixed time period. In other words, there is a statistically likely number of price bubbles of a certain magnitude range appearing within each time frame. Of course then, the higher the bubble magnitude, the lower the probability or the larger the time frame within which this bubble would be expected to appear. Consequently, by the central limit theorem the occurrence of the price bubbles of each magnitude is expected to approach a normal distribution under each magnitude range as the number of observations increases to infinity. This would be precisely the case with large market indices.

9 This conclusion can be drawn from feedback theory. The larger extent of the price bubble, the larger the number of investors attracted by potential profit opportunities. In this way, the extent of the initial bubble resonance translates into both the size and the life expectancy of the resulting bubble.

10 It can be speculated that yet another level of larger-scale asset bubbles is formed as a result of derivative markets with miscalculated security prices. Even the "rational" investors are expected to make their investments based on estimating risk and return through CAPM or the Black-Scholes formula, which implies that their risk estimates will inevitably be based on log-normal price distribution. As the lognormal price distribution tends to understate systematic risk [e.g. Mandelbrot (1963)]
3 Method

Now that the fractal model of a financial market has been suggested, it needs to be verified against its main rival, namely, the random walk. The purpose of this part is to determine and describe the mathematical models that need be combined in order to test a financial market for the above described fractal structure. After a test model is derived, the technical test hypotheses are defined for the subsequent empirical research. However, first a more intuitive metaphor of the above described fractal image of the market needs to be presented in order to clarify the transition from the implicit theoretical propositions above to the technical hypotheses that follow.

3.1 The “Boiling Water” Metaphor of the Market Fractal

The image of a financial market that emerges from the joint interpretation of Behavioural Finance and Fractal Theory can be described simplistically as an interaction between discretely distributed price bubbles, each of which is surrounded by chaotic and normally distributed price patterns and smaller price bubbles, while each of the smaller bubbles is in turn surrounded by yet smaller normally distributed and chaotic price patterns and bubbles etc. On the surface, such a process can indeed be mistakenly perceived as a pure random walk (imagine an asset bubble surrounded by white noise, chaos and smaller price bubbles with the same structure). However, it is essentially a fractal model whose mathematical properties differ substantially from those of a random walk process. In order to reveal the discrepancies, a more intuitive image of the fractal market structure needs to be presented.

In simple terms, the resonance between price bubbles in a financial market forms an image similar to water boiling in a cattle, with new bubbles constantly appearing and bursting and occasional drops flying out and flooding the surrounding area. A new bubble makes the water surface seem higher compared to its actual, or expected, level, while the explosion of a bubble creates a crater, making the surface seem lower.

Although this “boiling water” image is as seemingly random as the random walk model, the tools for potential analysis are drastically more abundant. The principal difference is that this "boiling water" metaphor implies a stochastic recurrence and interaction between similar oscillatory processes (bubble birth, bubble burst, a crater and finally an equilibrium) on multiple magnitude and frequency levels. The smaller bubbles and oscillations will generally occur at a higher frequency, while the larger bubbles are less probable and will thus occur more rarely. Interactions between each level of bubbles would create larger bubbles, giving the market index a fractal structure.

From an empirical perspective, this “boiling water” model can be tested by simply determining whether a large market index is separable into price bubbles of different magnitude and frequency ranges, with the residual noise being purely random on several magnitude levels. Thus, the principal difference from the random walk model is that the random walk hypothesis assumes that the percentage changes in prices are normal, while the “boiling water” metaphor assumes that the residual distribution of the price levels is normal on several scales once the effects of the price bubbles are eliminated. In this way, the “boiling wa-

---

they expect lower risk premia than would be advised under accurate risk estimations. The excess demand created by a systematic understatement of risk results in unsustainable excessive market capitalization. In the long run, this would drive the entire markets away from fundamental values, resulting in severe market corrections. It can be argued that this largely contributed to the market crash of 2008, where the new fixed-income instruments resulted in a major liquidity crisis.

---

11 The crater would represent the consequent deflationary bubble found in the simulation by Steglitz & O’Callaghan (see 2.2.3, Figure 2).
ter” model allows one to empirically test the predictions of Behavioural Finance and Fractal Theory against the predictions of EMH and CAPM expressed in the random walk model.

3.2 Related Empirical Research: A Note on Data Audification

An area of empirical study worth mentioning before proceeding with the model derivation is financial data audification. This method implies translating financial data into audio signals (acoustic representations), rather than plots against time (visual representations). The acoustic signatures are different for stochastic processes with different statistical properties. As a result, while it is generally impossible to distinguish between a simulated stochastic process and market data with a naked eye\(^{12}\), all is apparently not that hopeless when price movements are listened to instead.

Acoustic representations of various stochastic processes occur naturally. For example, the same white noise that represents the falsely assumed normal distribution of the returns could be heard at the background during early phone conversations. The acoustic white noise even became the foundation of the horror film ‘White Noise’ in 2005. The audification of initially non-audio stochastic processes allows the person to distinguish between the acoustic signatures of different stochastic phenomena and complex statistical distributions. Unfortunately though, as market price movements do not obey the exact same resonance laws as acoustic waves do [e.g. Frysinger (1990)], audification of the financial data is also a relatively complex process. For the most recent research on financial data audification, as well as a survey of previous sonification study, see Worrall (2010).

3.3 Frequency Domain Representation

In the case of the suggested “boiling water” model of the market index, each price bubble is essentially an oscillation around the expected value, which would correspond to a wavelet of a certain frequency and amplitude. Hence, the occurrence of a large number of price bubbles would generate acoustic noise, also of a certain frequency and amplitude. This “bubble noise” is expected to be different from the noise generated by a random walk. However, it would be nearly impossible to discern the overlapping bubble bursts from white noise by simply playing back the generated sound. This is why a more elaborate analytical tool is required to examine the “acoustic waves” generated by market index prices.

Because each range of bubble sizes is expected to correspond to a (finite) range of frequencies, analytically separating the frequencies should result in separating various ranges of bubbles, and thus eventually taking apart the fractal structure of the market index. In other words, in order to test for oscillatory frequencies in time series data, it is necessary to first make a frequency domain representation of the vector of the market index price observations.

The most popular mathematical method that performs a frequency domain representation of time domain data is Fourier transform, based on the work of a French mathematician Joseph Fourier (1768-1830). Fourier discovered a formula that could be used to transform any periodic function with a series of sinusoidal waves (i.e. sine and cosine functions)\(^{13}\). His work became the foundation of harmonic analysis, which is a generic term unifying several types of transforms and frequency domain representation techniques that use basic harmonic (sine and cosine) waves as building blocks. The principle of Fourier transform is essentially similar and complementary to the principle of audification (discussed under 3.1),

\(^{12}\) This observation can also be found in Mandelbrot’s ‘(Mis)behaviour of Markets’

\(^{13}\) A wavelet is a local occurrence of a periodic phenomenon. The simplest example would be the cardiogram of one heartbeat.

\(^{14}\) Fourier Analysis has been formerly applied in pricing European call options [Carr & Madan (1999)], and measuring stock market volatility [Malyavin & Manchino (2002)]. It has also been established that a Fourier series can be used to approximate the behaviour of a stationary random variable [Kosambi (1943); Karhunen (1947); Loeve (1978)].
as a sound wave is a classical example of a complex periodic function that can be represented in its frequency domain as a Fourier series.

3.4 Harmonic Analysis of a Financial Time Series: Introducing the OFS Transform

In the course of this section, a version of harmonic series representation is defined for a time series of market price observations. The series is given the name OFS, while the general method of its approximation is called the OFS transform. OFS transform incorporates the concepts of Fourier transform, OLS regression analysis, and optimization.

3.4.1 A Definition of the Fourier Series

Fourier analysis consists in a decomposition of a periodic (or stochastic) phenomenon as a series of sinusoid functions. A Fourier series of order “N” is formally defined as:

\[
(S_N, f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \{a_n \cos(nx) + b_n \sin(nx)\}
\]

This series is used to approximate a general periodic function \( f(x) \) with a period of \( 2\pi \). The coefficients \( a_n \) and \( b_n \) (inclusive of \( a_0 \)) are derived as follows:

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad n \geq 0
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \quad n \geq 1
\]

As “N” increases, the approximation becomes more accurate.

In separate cases, this modelling technique can be rewritten in an exponential form through some derivation of Euler’s formula:

\[
e^{ix} = \cos(x) + is\sin(x)
\]

3.4.2 Periodic Functions and OLS Regression Analysis\(^{15}\)

A periodic function such as a sinusoid can be transformed in such a way as to resemble most simple functional forms used in OLS regression analysis. To give an example, the most commonly used OLS regression functions have a linear (a) or some type of a logarithmic (b) form:

(a) \( Y_i = \beta_1 + \beta_2 X_i + \epsilon_i \)

(b) \( \log(Y_i) = \beta_1 + \beta_2 X_i + \epsilon_i \) (a log-lin function)

Both of these functions can be approximated by transforming sinusoid functions. Case (b) can be easily rewritten as \( Y_i = e^{\beta_1 + \beta_2 X_i} + u_i \), which is transformable into a periodic expression by Euler’s formula men-

\(^{15}\) There has been prior research on estimating a Fourier series on discrete data through OLS regression techniques. The corresponding Fourier series is called the Regressive Discrete Fourier Series (RDFS). However, the original RDFS formulation and its estimation methodology [Arruda (1992a, 1992b)] differ from the model and estimation methodology suggested in this paper.
tioned above. Taken case (a), the linear function can be approximated by infinitely stretching the portion of the sinusoid function at the point where \( \frac{d \sin(x)}{dx} = \beta_1 \), and then shifting the resulting line vertically by \( \beta_1 \).

A simple sinusoid function can approximate \( \beta_2 \) coefficients of up to 1. Linear functions with larger \( \beta_2 \) coefficients can be approximated by scaling the sinusoid function with either an amplitude parameter \( p \), or a frequency parameter \( q \), or both:

\[
\frac{d[p \sin(qx)]}{dx} = \beta_2
\]

The expression above will approach infinity at \( x=0 \) for large values of \( p \) or \( q \).

### 3.4.3 Estimating “p” and “q” through Optimization: The OFS Defined

The conventional way of estimating a least-squares function includes a manual choice of functional form, followed by a manipulation of the variances and covariances of the random variables. However, since a transformation of the sinusoid function can be used to approximate a range of popular least-squares functions, it is possible to implement an algorithm estimating the sinusoid transformation that would best satisfy the least-squares condition, thus automatically approximating the optimal functional form for regression analysis.

There are three types of transformation for a sinusoid function, which include a vertical shift (denoted here as \( C \)), a horizontal shift (here, \( h \)), an amplitude parameter (here, \( p \)) and a frequency parameter (here, \( q \)). A general form of a simple periodic function \( y \) of a variable \( x \) can thus be written in the form:

\[
y = p_1 \sin(q_1 x + h_1) + p_2 \cos(q_2 x + h_2) + C
\]

If \( x \) and \( y \) are random, the function used to estimate the relationship between \( x \) and \( y \) should be:

\[
y_i = p_1 \sin(q_1 x_i + h_1) + p_2 \cos(q_2 x_i + h_2) + C + \epsilon_i,
\]

where \( i \) would correspond to the \( i \)'th observation of \( y \) and \( x \). Alternatively, if \( y \) were a time-series variable, its behaviour could be described as a function of time \( t \):

\[
y_t = p_1 \sin(q_1 t + h_1) + p_2 \cos(q_2 t + h_2) + C + \epsilon_t, \text{ where } t = 0, 1, 2, 3, \ldots
\]

If we solve the above equation for the error-term and apply the least-squares condition, we can formulate the objective function:

\[
\sum_{t=1}^{n} \left( y_t - \left[ p_1 \sin(q_1 t + h_1) + p_2 \cos(q_2 t + h_2) + C \right] \right)^2 = \min
\]

This is a programming problem that can be solved by setting \( C = \frac{\sum_{t=1}^{T} y_t}{T} \) and altering \( p_1, q_1, h_1, p_2, q_2, h_2 \). These parameters can be subjected to constraints to simplify the optimization process:

\[
\begin{align*}
C &= \frac{a_0}{2}; \quad q_1, q_2 = n; \quad p_1, p_2 = b_1, a_1; \quad h_1, h_2 = 0
\end{align*}
\]

\[16\] Note that this expression gives a Fourier series of order 1, as long as:

\[
C = \frac{a_0}{2}; \quad q_1, q_2 = n; \quad p_1, p_2 = b_1, a_1; \quad h_1, h_2 = 0
\]
1. \( h_1, h_2 \in \left[ -\frac{\pi}{2}; \frac{\pi}{2} \right] \): otherwise, the cosine and sine terms are interchangeable.

2. \( p_1, q_1, p_2, q_2 > 0 \): identical effects can be achieved by altering and \( h_1 \) and \( h_2 \).

As can be seen, the optimization-based estimation of \( y = p_1 \sin(q_1 t + h_1) + p_2 \cos(q_2 t + h_2) + C + \epsilon \) is not completely identical to the differential method suggested in the original Fourier transform. More precisely, the above defined series of harmonic functions is an approximation of a time series dataset for which a mathematical expression is not defined. This is why it would be appropriate to give the above approximation the generic name Optimal Fourier Series, or OFS for short.

Part of the reason for this discrepancy is that there is by definition no readily available functional expression for the behavior of the financial data. However, the generic chaotic process arguably followed by the market index prices does theoretically have a mathematical description.

### 3.4.4 An OFS of Order "N", and the "Break" Parameter

Up until now, the discussion concerned approximating a financial time-series process with an OFS of order 1. However, as was mentioned in 3.1, the Fourier approximation becomes more accurate when the number of terms is increased. This can be achieved through repeating the optimization process from 3.3.3 a number of times, using the residuals from the previous regression as the observation vector. All optimization stages after stage 1 will not require a vertical shift parameter, as the mean error will be 'zero' (this is achieved due to the mathematical properties of the OLS condition).

It is possible to try and approximate the entire random vector through a Fourier series, but this can result in an infinite number of terms and thus an infinite optimization process. A plausible alternative would be to only estimate an OFS series of a certain maximum order, or to stop at approximate normality in the residuals. In a programming vinacular, this type of a constraint would be called the "break" parameter.

While defining the maximum number of terms is straightforward, a normal residual distribution is a more complex parameter to define. Normality can be tested in several official ways, and each of these tests may or may not be regarded as a sufficient indicator of normality in the residual observation vector. In the current research, the algorithm will rely on the Jarque-Bera test [Bera & Jarbue (1980)], formally defined as:

\[
X = \frac{n}{6} \left[ S^2 + \frac{1}{4} (K - 3)^2 \right],
\]

where \( n \) is the number of observations, \( X \) follows a Chi-squared distribution with 2 degrees of freedom, and:

\[
\sin(-x) = -\sin(x); \quad \sin(x + \frac{\pi}{2}) = \cos(x); \quad \sin(x - \frac{\pi}{2}) = -\cos(x); \quad \sin(x + \pi) = \sin(x - \pi) \nonumber \\
= -\sin(x); \quad \cos(-x) = \cos(x); \quad \cos(x + \frac{\pi}{2}) = -\sin(x); \quad \cos(x + \pi) = \cos(x - \pi) \nonumber \\
= \cos(x - \pi) = -\cos(x); \quad \sin(x) + \cos(x) = \sqrt{2}\sin \left( x + \frac{\pi}{2} \right)
\]
\[
S = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^3 \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \right\}^{-\frac{3}{2}}, \quad K = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^4 \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \right\}^{-2}.
\]

In statistical terms, \( S \) denotes the skewedness of the probability density function, i.e. its asymmetry with respect to the maximum likelihood point (i.e. the maximum point on the PDF). \( K \) denotes the kurtosis, which is the relative weight of the outliers in the distribution.

As a normal distribution has a skewedness of '0' and a kurtosis of '3', \( X \) tests \( S \) and \( K \) jointly for being, respectively, '0' and '3'. A value below 0.05 in the Chi-squared distribution function with 2 degrees of freedom with respect to the value of \( X \) would signify that the null hypothesis of a non-normal distribution is rejected at the 5% level. This test is, however, sensitive to small data samples, which limits the realm of application of the current approximation to the relatively large data samples.

3.5 Defining the Research Hypotheses

A technical comparison between the fractal model and the random walk model necessitates a set of testable hypotheses that can be supported or rejected by statistical methods. This paper will concentrate on two main research hypotheses, both of which follow directly from the fractal model of the financial market and are thus central to the fractal theory argument. These two hypotheses can be defined as follows:

**Fractal Market Hypothesis 1:** In a large market index, price movements on every magnitude and frequency scale can be separated into two components: price bubbles and residual noise. When the price bubble component has been separated, the residual noise will be approximately normal. The more levels of price bubbles have been separated, the more will the residual noise approach a normal distribution.

**Fractal Market Hypothesis 2:** Despite its partial resemblance to random processes, the proposed fractal market structure will exhibit deterministic patterns on a large observation scale. If financial markets exhibit the proposed fractal structure that only resembles a random walk process, the out-of-sample predictions of the OFS model are expected to be more accurate compared to the out-of-sample predictions of the random walk model.
4 Results and Discussion

Under this title, the results of applying the OFS model to a market index are presented. This paper uses the daily high-low average of the S&P500 index for the period between January 1, 2001 and December 31, 2010, comprising a total of 10 years of daily observations. For the subsequent out-of-sample forecasting tests, the scope of observation is extended by three months, i.e. to March 31, 2011. The high-low averages were taken instead of the conventional closing prices to eliminate the random errors inherent in the closing price observations.

The data is taken from finance.yahoo.com and is publicly available. The number of observations in the original sample (excluding the missing values and the forecasting period) is 2514. After the missing dates have been added as gaps (‘NaN’) between the price quotes to restore the original seasonality, the resulting number of observations including the missing values has increased to 3652. The number of observations including the forecast period is 2577 and 3741 observations, respectively.

All historical price data is necessarily discrete. OFS transform assumes by its nature that the discrete historical price observations represent a sample from a continuous-time process\(^ {18}\). In this way, the presented approximation incorporates a probabilistic guess as to the behaviour of the S&P500 stock prices beyond the scope of in-sample observation. Figures 5-7 below gives a graphical representation of the approximation results, together with the actual observations and the residuals. The R-squared coefficient of the estimated OFS is 98.1%. The derived coefficient estimates can be found in Appendix 2.

\[\text{Figure 5: an OFS Approximation of a 10-Year S&P500 Price Series plotted against a time trend}\]

\[\text{\textsuperscript{18} In this way, the OFS method resembles the RDFS method. One potential benefit is an ability to predict the expected (instantaneous) rate of change in the stock prices even when the actual observations are unavailable.}\]
It can be seen with a naked eye that the fitted OFS values in Figure 5 account for a substantial portion of the price movements in the S&P500 index. The R-squared coefficient of 98.1% suggests that only 1.9% of the variation in the S&P500 prices remains unexplained by an OFS of order 208. The Q-Q-Plot of the residuals is congruent with a nearly perfect normal distribution (Figure 6), which is also supported by the Jarque-Bera prob.-value of 0.0001 (Figure 7).

The behaviour of the Jarque-Bera test statistics and probability values as more OFS terms are added (Figure 7) is what would be implicitly expected under Hypothesis 1. As multiple price bubbles can be expected to occur on each magnitude and frequency, an OFS series of a high order may be required to model (and thus eliminate) the effects of these bubbles on the market price levels. Hence, the movement of the Jarque-Bera test statistics can also be expected to follow a cyclical pattern, with a long-term tendency to decline as ever more bubbles are taken out of the picture. In other words, in either of the graphs in Figure 7,

A Quantile-Quantile-Plot (Q-Q-Plot) is a plot of the empirical probabilities of the elements of a random vector, against the theoretical probabilities of the same elements under a normal distribution with an identical mean and variance. The identity line represents the relationship that is expected to arise when a given random vector follows a perfect normal distribution.
each cycle with an increase followed by a sharp decline can be interpreted as one scale of price bubbles being approximated by OFS transform and subtracted from the market price pattern. It can be seen from the plot of probability values in Figure 7 that approximate normality has been achieved by OFS term 180, while the subsequent 208 terms reduced the price movements to nearly perfect white noise.

It can be argued that, despite some statistical evidence to the contrary, the residuals in Figure 5 exhibit serial correlation. Figure 10 reproduces the estimation residuals from Figure 5 to present a higher-resolution picture:

![Figure 8: OFS Approximation Residual Plot](image)

The seasonality that can arguably be observed in the residuals is in no way evidence against the fractal theory hypothesis. On the contrary, the observable residual oscillations in the residual stock price movements of the S&P500 index can be considered as evidence of a recurrence of its fractal structure on a smaller scale. At the same time, the statistical evidence that the residuals above are normally distributed suggests that at this level of price observation, the random walk model (or possibly some other simple time series model) would yet again be a “fair” approximation of stock price behaviour.

In general terms, Hypothesis 1 can not be rejected, as strong statistical evidence has been found in support of the predicted fractal structure of the S&P500 index.

### 4.1 Forecasting

In light of the models used for technical and fundamental analysis of the exchange rates, this paper’s empirical results beg the question: ‘Does this model fit out of sample?’ [Meese & Rogoff (1983)]. Indeed, the random walk model owes its popularity, at least in part, to the general failure of complex technical analysis to outperform complete randomness in out-of-sample predictive capacity. This is why a comparison between the predictive capacities of the OFS model and the random walk model may be a valuable tool in comparing the assumptions underlying the EMH with the assumptions underlying Fractal Theory and Behavioural Finance.

This section presents a comparison of the out-of-sample forecasting capabilities of the random walk model and the derived OFS series. The forecasting period is set at three months, ranging between January 1 and March 31, 2011. Figure 8 below plots the OFS forecast values together with the actual price observations. Figure 9 plots the absolute errors of the OFS forecast and the absolute errors of the random walk forecast against time.
Figure 9: Actual and forecasted S&P500 price levels plotted against a time trend. The vertical line at observation 3652 denotes the beginning of the out-of-sample forecast.

Figure 10: Absolute out-of-sample forecast errors of the OFS and random walk models

The conventional out-of-sample prediction error\textsuperscript{20} ratios can be summarized as follows:

Table 1: Forecast error ratios of the random walk and OFS models

\begin{tabular}{l c}
\hline
Forecast error ratios of the random walk (RW) and OFS models: \\
\hline
Mean Error Ratio (RW/OFS): & -17.9515 \\
Mean Absolute Error Ratio (RW/OFS): & 18123 \\
Root Mean Squared Error Ratio (RW/OFS): & 17545 \\
\hline
\end{tabular}

In other words, over the observed three-month out-of-sample forecast period, the mean error is nearly eighteen times higher, the mean absolute error is 1.81 times (or 81\%) higher, and the root mean squared error (RMSE) is over 1.75 times (75\%) higher with the random walk model than with the OFS model.

The results summarized in Figures 8 and 9 and in Table 1 all favour the OFS approximation as a superior out-of-sample forecasting tool compared to the random walk model. The OFS model outperforms the random walk model significantly by all three measurements of forecasting error. Figure 8 graphically demonstrates the out-of-sample predictions of the OFS model over a three-month time horizon. It can be argued that the OFS predictions follow the actual market prices to a remarkable extent. The comparative

\textsuperscript{20} For a discussion of the mathematical meaning of the three error measurements, see Meese & Rogoff (1983).
plot of the mean absolute errors of the OFS and random walk models (Figure 9) also demonstrates intuitively that the forecasting errors of the OFS model are on average significantly lower than the errors of the random walk model. Despite a small period of observation, it can also be inferred that the advantage of the OFS model does not diminish with time.

In other words, Hypothesis 2 cannot be rejected as consistent statistical evidence shows that the OFS model outperforms the random walk in out-of-sample forecasting capabilities.
5 Limitations and Conclusions

This paper’s empirical tests seem to provide consistent and statistically powerful results. At the same time, there are also a number of considerable limitations. For example, OFS transform is of course a simplistic model to test for a fractal pattern. In general, the proposed fractal market hypothesis implies perfect cyclicity on an infinitely large scale of observation. Under this assumption, an OFS would be a perfect approximation of an infinitely large market index. However, once brought to such an extreme, the current fractal market hypothesis becomes essentially as unrealistic as the random walk hypothesis. An accurate approximation of the fractal model of a real market index would thus necessitate an introduction of a chaotic (path-dependent) component into the OFS model. One possibility to accomplish this goal is discussed in Appendix 1, section 5.1.2 (GARCH and Variable OFS Coefficients).

In terms of the empirical results, the amount of data is not sufficient to make definitive conclusions about the merits of the OFS model. Ideally, out-of-sample forecasts would need to be performed over a longer time horizon, while the forecast errors may need to be measured against more complex time series models, such as GARCH or ARIMA processes. The close relationship observed between white noise and chaos also points at a possible endogeneity problem, implying that OFS transform potentially needs to be applied to a simulated random walk process. In that way, it can be verified with a higher degree of certainty that the resemblance of a random walk is indeed an endogenous byproduct of the exogenous general chaos.

Additionally, as with all statistical research methods, there are weaknesses and limitations inherent in the mathematical method underlying OFS transform. Admittedly, any type of harmonic analysis assumes by its design that the time domain data has naturally occurring oscillatory properties on multiple frequency scales. In other words, testing for cyclicity by means of a Fourier-based transform necessarily results in a high danger of a circular argument and a resulting type II error, implying that Fourier transform can not be used to test the cyclicity hypothesis without powerful alternative grounds on which the same cyclicity hypothesis could be rejected. This paper has limited the critical evaluation of the OFS analysis to an investigation of its forecasting capabilities as compared to pure randomness. This method is a valid remedy against spurious results, but arguably less so against alternative systematic models of a financial market.

Despite these and other limitations inherent in the current research, the obtained test results are statistically powerful and consistently supportive of this paper’s view of Behavioural Finance and Fractal Theory as two complementary conceptual frameworks that can rival the classical finance theory. As such, the empirical findings support the view of a financial market as a complex fractal structure that arises from a complex network of chaotic processes. At the very least, this new theoretical approach to market analysis appears to incorporate considerable potential for further development.

As for the general merits of the current research, this paper seems to have managed to demonstrate with an arguable degree of success, both in theory and in practice, the possible path of the transition from the local chaos of human behaviour to the randomness of security prices, and from the randomness of the security prices to the ordered nature of a financial market. This synthetic approach to market analysis can be viewed as a step towards reconciling Behavioural Finance and Fractal Theory into a unified framework that can potentially defend the local imperfections of human behaviour against the restrictive assumptions of rationality and efficiency.
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Appendix

Appendix 1: Further Research Prospects

This additional section lists the potential areas of improvement for the derived empirical method. Some of the improvements described here will be considered in detail in the forthcoming research by the same author.

5.1 Applying OFS to Longer-Period Historical Data

The current method of Fourier transform can be argued to only apply during the time periods where the price bubbles and price oscillations around the rational expectations dominate the long-term (exponential) trend of market capitalization at the rate of return on equity. It can also be argued that the general volatility of the market has increased dramatically in the course of the past several decades (see Figure 1I).

Figure 1I: Historical S&P500 prices over the past 60 years. Source: finance.yahoo.com

These dominating macroeconomic trends potentially dismiss simple Fourier transform as a valid tool for technical analysis, even if the Fractal Theory and Behavioural Finance assumptions actually hold. As seen with a naked eye, there are two possible solutions to this problem, both of which are discussed below.

5.1.1 Correcting for Market Capitalization

A simplifying adjustment can be made about the rate of market capitalization in the long run. In particular, one could assume that in the long run market growth is continuous and exponential at a relatively constant rate. Accordingly, price oscillations and bubbles would potentially occur around the deterministic trend of market capitalization defined as “\( y_t = c + e^{rt} \)”, where \( y_t \) denotes market capitalization at time \( t \), \( e \) is Euler’s constant and \( r \) is the average rate of return that can be estimated by simple OLS methods. Additionally, it can be speculated that the rate of volatility increases proportionally to the level of market capitalization\(^\text{21}\).

The Fourier series optimization problem defined in 3.3.3 can be then redefined for application to financial market data over a long time horizon. The new equation would have the form of a following nature:

\(^{21}\text{Ironically, this assumption is consistent with the Geometric Brownian Motion model of the stock prices, which is in turn consistent with the random walk hypothesis.}\)
\[ y_t = c + e^{rt}[1 + p_1 \sin(q_1 t + h_1) + p_2 \cos(q_2 t + h_2)] + \varepsilon_t \]

What is different in the equation above compared to the original equation in 3.3.3 is, the intercept \(c\) and amplitude terms \(p_{1,2}\) are now determined to increase exponentially with time. The frequency of oscillations is still assumed to be constant which can be, however, corrected with an additional minor improvement to the formula if an exponentially increasing oscillation frequency were favoured by data evidence.

### 5.1.2 GARCH and Variable OFS Coefficients

It has been established in econometric research that volatility in the financial markets is both autoregressive and correlated with prices and returns. In other words, an increase in volatility at time \(t\) will be reflected in both the volatility at time \(t+1\), and the \(t+1\) price levels. The models that consider this relationship are called the 'Variance Function Models', the most popular of which is the GARCH model:

\[ y_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \sigma_t \quad \text{is the autoregressive variance of} \quad y \quad \text{defined as} \]

\[ \sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2} \]

In fact, there exist several variance function models based on GARCH, some of which use non-linear functional forms (e.g. E-GARCH), assume an infinite variance (I-GARCH), or include conditional standard deviation as one the regressors (GARCH-M).

GARCH processes can be integrated into the constrained optimization problem defined in 3.3. For that purpose, the amplitude of the change in a financial time-series variable has to be defined as a function of some volatility measure, such as past conditional standard deviation or the absolute past errors. This implies variable amplitude and frequency coefficients \(p_1, q_1, p_2\) and \(q_2\). From now on, capital letters will be used for the optimization coefficients that are functions of volatility, and small letters will be used for constants and variables.

A generalized optimization coefficient \(K\) at time \(t\), with a constant and a stochastic (GARCH) term would be best defined by:

\[ K(v_1, v_2, ..., v_n) = k_0 \left\{ k_1 + \sum_{i=1}^{n} \sum_{j=1}^{m} k_{i+1,j} v_{i,t-j} \right\}, \quad k_{i,j} \in [0; 1] \]

By assigning different expressions to \(v\), it is possible to estimate different GARCH-based models in the calculation of \(K\). Introducing this method of coefficient estimation into the optimization formula from 3.3.3 would yield:

\[ y_t = c + p_1 \sin(Q_1 t + h_1) + p_2 \cos(Q_2 t + h_2) + \varepsilon_t, \quad \text{where} \]

\[ p_1 = p_{1,0} \left\{ \frac{p_{1,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} p_{1,i+1,j} v_{i,t-j}}{p_{1,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} p_{1,i+1,j} v_{i,t-j}}, \quad p_{1,i,j} \in [0; 1], \right. \]

\[ p_2 = p_{2,0} \left\{ \frac{p_{2,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} p_{2,i+1,j} v_{i,t-j}}{p_{2,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} p_{2,i+1,j} v_{i,t-j}}, \quad p_{2,i,j} \in [0; 1], \right. \]

\[ Q_1 = q_{1,0} \left\{ \frac{q_{1,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} q_{1,i+1,j} v_{i,t-j}}{q_{1,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} q_{1,i+1,j} v_{i,t-j}}, \quad q_{1,i,j} \in [0; 1], \right. \]

\[ Q_2 = q_{2,0} \left\{ \frac{q_{2,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} q_{2,i+1,j} v_{i,t-j}}{q_{2,1} + \sum_{i=1}^{n} \sum_{j=1}^{m} q_{2,i+1,j} v_{i,t-j}}, \quad q_{2,i,j} \in [0; 1]. \right. \]
Appendix

A GARCH coefficient estimation in the optimization problem 3.3.3 has the potential to eliminate the varying volatility problem in long-run market index data. Additionally, an inclusion of the GARCH component into the optimization problem can potentially drastically reduce the number of terms required for a frequency domain representation of the market index prices. It may also be a superior tool for in-sample and out-of-sample forecasting. Optimization-based Fourier transform with the GARCH component will be introduced and applied to financial time series data in a subsequent paper, with a detailed comparison with the (present) constant-coefficient OFS model.