Making distinctions – critical for the learning of the 'existence' of negative numbers? Exploring how 'instructional products' from a theory informed Lesson study can be shared and enhance student learning

Ulla Runesson Kempe, Anna Lövström and Björn Hellqvist

School of Education and Communication, Jönköping University, Sweden; ulla.runesson@ju.se

In this paper, we present how experiences gained from a theory informed Lesson study in regards to a specific learning goal can be shared and used by other teachers in new contexts. A group of teachers worked together in a cyclic, iterative process of planning, evaluating and revising teaching. The aim was to provide possibilities for grade 2 and 3 students to become familiar with negative numbers, thus to extend the number range from N to Z. Based on theoretical tenets about learning and empirical findings in the study the teacher group draw the conclusion that the pupils needed to be able to differentiate some aspects of negative numbers. The conjecture was put to the test in a follow-up study with five new teachers and eight classes. One lesson was taught based on the empirical finding in the Lesson study. When learning gains from pre- to post-test in these classes were compared to those in the Lesson study, similarities were found. It is suggested that accumulate evidence about 'what must be learned' across different classroom settings can be gained through a theory informed and goal oriented Lesson study.

Keywords: Lesson study, Learning study, teacher collaboration, sharing instructional products, learning and teaching negative numbers, variation theory.

Introduction

Morris and Hiebert (2011) have presented Lesson study as a model for transforming teachers' craft knowledge into professional knowledge i.e. making it public, sharable, storable and verified as well as improved and should be organized around public changeable knowledge products. Their arguments are based on the necessity to reduce differences in classroom instruction (ibid. p.5). The aim of this paper is to illustrate and discuss what an instructional product, generated on the basis of a pedagogical theory and empirically grounded, could look like and whether making use of such a product can be productive to enhance student learning. We report on how a theory-framed version of lesson study – Learning study – can produce 'instructional products' useful outside the specific context. Insights gained from one Learning study (LrS) about how to enhance the learning of negative numbers were communicated and used by new teachers in new contexts.

Lesson study not just professional development

Stigler and Hiebert (1999) pointed out the effectiveness of Japanese Lesson study (LS) model for improving teaching and learning of mathematics. There are extensive reports on the effectiveness of Lesson study for teachers' improvement of teaching skills; how they learn to reflect, on changes in motivation and capacity to improve instruction and the development of content and pedagogical content knowledge (e.g. Lewis, Perry & Hurd, 2009,). Furthermore, it is often pointed out how to how Lesson Study can promote the establishment of learning communities and teacher collaboration, a culture of mutual accountability, shared goals for instruction and a common language for analyzing instruction (e.g. Chichibu & Kihara 2013; Hunter & Back, 2011¸ Toshiya & Toshiyuki, 2013.) To us, with these purposes, Lesson study will be restricted to a model for professional development only, not as a system that can generate new and relevant knowledge recognized as a legitimate knowledge source for professionals.

Hiebert and Morris (2011) takes Lesson study further when they promote it as a system for "the creation of shared instructional products that guide classroom teaching" (p. 5). 'Instructional products' should be designed with a specific learning goal in focus and detailed enough to guide classroom instruction. An instructional product is the current answer to common and shared problems on teaching and learning. It is tentative, changeable and thus, open to improvement. Therefore, such 'local theories' embedded in the instructional product must be communicated, shared and improved by other teachers in other contexts. In this way, they could be tested and verified under new and local conditions.

Learning study

Learning study is a theory-informed version of Lesson study (Marton & Pang, 2003). It shares features with Lesson study, such as the collaboration among teachers and the iterative design of planning, implementing, observing and revising of the lesson, but is framed by a theory of learning —variation theory (Marton, 2015). Just as with Lesson study, there are reports on the positive effects of Learning study on teachers' professional development (e.g. Lo, Chik and Pang, 2006). Learning study is also a model for constructing knowledge concerning the objects of learning as well as the teaching-learning relationship. It takes the professional task as the point of departure and generates public and sharable knowledge for the improvement of teaching and learning, exactly in line with what Hiebert and Morris (2011) advocate.

The knowledge produced in Learning study is an instructional product, not in terms of a lesson plan or specific teaching methods, but in terms of what is found to be necessary to learn in order to develop a specific understanding, skill or attitude. It is not about learning in a general sense, but in relation to specific learning goals (cf., Hiebert & Morris, 2011). In Learning study, variation theory serves as a tool for teachers to identify the necessary conditions or learning the object of learning. Learning, from this theoretical perspective, is seen as a change in one's way of experiencing something. How we experience something has to do with what aspects we notice and become aware of. For every object of learning there are certain critical aspects necessary to discern.

'Critical aspects' are dimensions of variation in the object of learning that the learner has not yet learned to discern and attend to. It has been suggested, however, that the critical aspects must be identified for every group of learners (Pang & Ki, 2016). Variation theory takes a relational perspective on learning, meaning that the critical aspects are not merely a feature of the content (a concept for instance), but a feature of the experienced object of learning. They cannot be derived at "from disciplinary knowledge alone or as taken-for-granted truths" (p. 333) Learners bring various experiences to the classroom and experience phenomena in different ways. Therefore, to identify the critical aspects, learners' ways of experiencing must be taken into account. In Learning study, this is done by carefully diagnosing—via interviews and/or written tests—before and after the lesson. So, ‘critical aspects’ should be defined in relation to the phenomenon in question as experienced by learners rather than in relation to what is deemed critical in the curriculum or subject discipline (ibid. p. 328)

Marton (2015) asserts that one cannot become aware of new concepts or aspects without becoming aware of differences (i.e. variation). Variation theory is used when the teachers explore students’ prior understanding and to what extent the object of learning has been achieved by the learners after instruction. The exploration of teaching and learning in the learning study entails identifying what aspects of the object of learning that are critical for learning and how to make it possible for the learners to experience them. When planning the lesson, variation is used for creating problems, example spaces and choosing representations for example.

Teaching and learning negative numbers—some recommendations

Gaining understanding of the nature of negative numbers has been problematic for early mathematicians to comprehend (Bishop et al., 2014), as well as for teachers to teach and learners to learn (e.g. Ball, 1993). The difficulties have to do with the meaning of the numerical system and the magnitude and direction of the number, the meaning of arithmetic operations, and the meaning of the minus sign (Altiparmak, & Özdoğan, 2010). For Swedish students the meaning of the minus sign is probably particularly difficult, since in Swedish a number like –2 (in English: negative two) is pronounced as ‘minus två’ (minus two) and written –2. Thus, there is no linguistic and symbolic difference between the minus as a sign for the operation and as a sign for the number.

It has been recommended that teaching of negative numbers should take the point of departure in real life problems or situations known from the children’s experience and transformed into mathematical models. For instance, using ‘a house’ with floors above and below the ground floor, or a bird flying/diving above/below sea level, has been suggested (Ball, 1993). Usually in the Swedish mathematics curriculum negative numbers are contextualized within discussions about temperature below and above zero and with the help of the thermometer. However, there might be a risk with this. The number system and the ordering of integers might not be visible when negative numbers are talked about as ‘minus-degrees’ (in Swedish: ‘minus-grader’). Every child probably knows that it is colder when the temperature is –10 degrees C compared to a temperature of 3 degrees C. This may be confusing when they have to learn that –10 is a smaller number than 3. This was also found initially in the Learning study reported here. So, the teachers decided to use the number line only and talk about the numbers within a mathematical context instead of referring to o temperature or depths.

A Learning study on expanding students’ number range from N->Z

In the Learning study (LrS) one of the authors of this paper worked in collaboration with two primary school teachers and 64 students in four different classes in grade 2 and 3 (8–9 years old) in Sweden. The teachers wanted to extend the students’ experience of numbers to include the negative numbers also. In doing so, they explored what the students must learn—thus finding the critical aspects—in order to be familiar with integers and how to teach this in a way that would enhance the students’ learning.

The LrS encompassed four cycles, that is, four lessons were taught with four different classes. A diagnostic pre- and post-test was given to the students. Results from this, together with a close analysis of the recordings of lesson, gave insights into what is critical for learning and how the content must be handled to promote learning. Thus, when the learners failed to learn that which was targeted, they had to go deeply into the lesson and inquire how the content was handled and whether it was made possible to learn that which was intended. This analysis became the basis for the planning of the following lesson in the cycle, which was taught by a new teacher, and to *new students*, and again the recorded lesson and the diagnostic post-test are analyzed. The iteration proceeded until all classes were taught. Hypotheses about the critical aspects were tested in class. So, the critical aspects emerged as a result of trying them out in class and carefully analyzing students' learning outcomes and what was made possible to learn in the lesson. When it was found that the learning outcomes were not as expected, the teachers had to consider the possibilities for learning during the lesson and, by being guided by variation theory, discuss learning in terms of discernment. As the process continued the critical aspects became more specified; from something to be discerned, to something that should be differentiated, namely:

- To differentiate the value of two negative numbers
- To differentiate the function of the minuend versus the function of the subtrahend in a subtraction
- To differentiate the minus sign for negative numbers versus the minus sign for subtraction

To get the students to discern the critical aspects, carefully constructed examples, based on the idea of variation/in-variance were used. So, for instance, the examples $3-2=$ and $2-3=$ ('3' varies; minuend/subtrahend) were contrasted as operations on the number line. The choice and character of the example space (Watson & Mason, 2006) was changed and developed during the process. It was not until lesson 4 that examples like $2-4=$ and $-2-4=$ were implemented in the lesson, for example. Since the results on the post-test after lesson/class 4 was were significantly better compared to the previous lessons, it was concluded that the examples chosen and how they were sequenced seemed to be important for the possibility to discern the critical aspects.

Putting the conjecture to the test: the follow up study

Lövström (2015) concludes that when the critical aspect was phrased in terms of differentiation, that is what things could be compared, it indicated not just what dimension that must be opened up, but also what values in that dimension that were critical and needed to be contrasted (two or more negative numbers). Thus, critical aspects in terms of differentiation highlight a specific subject matter and student's experience of the content, and furthermore, give direction to handling of the content.

To put the conjectures of the critical aspects identified in the LrS to the test, a follow-up study (FS) with eight classes of new learners (N=116) and five (partly) new teachers were conducted. All the teachers had more than 15 years of teaching experience. All except one, were primary school teachers and thus, not specialized in mathematics. Three of the teachers taught two classes each. One of them had participated in the LrS and is one of the authors of this paper. All except one were, to a varying extent, familiar with variation theory. The teachers were selected on the basis of previous interest in Learning study and variation theory and asked to teach one lesson (three of the teachers in two different classes) about negative numbers. One of the classes was grade 7, a group of learners with difficulties in mathematics, all the other were grade 2 and 3. Swedish was the first language for the majority of the students, but several other languages were represented in all classes. The guardians had given their written consents to student participation. The students were given a test (with a few exceptions identical to the test in LrS) before and after the lesson.

The FS was planned in a 3-hour meeting with the teachers and two of the authors of this paper. Results from the pre-tests in the eight classes were presented and discussed and it was found that the 'new' group of students had similar problems to the students in LrS. So, the critical aspects identified in LrS, were assumed to be valid for the new group of students also. The results from LrS were presented to the teachers and the identified critical aspects were described and discussed. A video-recording of lesson 4 was observed by the group. Some sections were repeatedly paid attention to. It was specially observed and discussed in detail how the number line was used in the lesson. The aim was to conduct the eight lessons as similar as possible in terms of how the critical aspects were handled. Similar, it was important that all the examples presented and discussed in lesson 4, were present in all the 'following up-lessons' just as the usage of the number line. Except for these requirements, the teachers were free to arrange the lesson in their own fashion; to choose group- or individual work, for example.

The FS did not have the same iterative design as the LrS. It was conducted in parallel during the same week. The lesson was conducted mainly in whole class, intersected with individual and/or group-work. The interaction was more of a discussion between the teachers and the students with probing questions around the examples presented on the board. The examples used opened up dimensions of variation and were designed to make the critical aspects possible to discern. The teacher drew the learners' attention to differences in the midst of similarities and the students were required to justify their answers, sometimes after discussion in peers/groups. The lessons lasted about 60 minutes. In our experience, in Swedish schools it is uncommon that such a long period of time is allocated to whole class teaching of mathematics among younger students. Still, the students seemed to concentrated and focused.

The data consists of video-recordings of eight lessons, and results on four tasks (1, 3, 4 and 9) in the pre- and post-tests. Here, only results from pre- and post-test are drawn on. In task 1 (a–e) students should choose the biggest of five numbers. In a) all were positive numbers, b)-c) negative and positive numbers and zero, and e) negative numbers only. The object of learning was not preliminary to operate with negative numbers, but in order to test if the students were able to experience that there are numbers < 0, operations with negative difference were chosen. So, task 3 (8 items) was subtractions with positive or negative difference. The subtrahend and the minuend were positive numbers except in g) where the subtrahend was negative. Similarly, task 4 was a subtraction with negative difference. Here the students should also give a justification of their answer. Task 9 was about the difference of the meaning of the minus sign. The test comprised another four tasks not accounted for here, however.

Some preliminary results based on a measurement of correct answers on the pre- and post-test are presented here. Results on the pre-test were compared to the post-test on each task and on a group level. A comparison between the LrS-group and the FS-group was also made. Preliminary results are presented in this paper.

Results

Preliminary results from the analysis of two tasks for all the groups are presented in Table 1 and 2.

Table 1: Numbers (percentage) of students who answered correctly on task 1, ordering of numbers

Item	a	b	c	d	e
Pre-test	113(97)	84(72)	104(90)	65(56)	34(29)
Post-test	112(96)	102(87)	112(97)	97(84)	85(73)

N=116

On In task 1 there were learning gains in terms of numbers and percentage of students who displayed the targeted experience of integers on all except one item. As can be seen from Table 1 the frequency of correct answers was higher on all items on the post-test, expect for a) (which had a high rate from the beginning). The highest increase is on d); from 56% to 84% and e): from 29% to 73% who answered correctly. Item d (negative numbers and zero) and particularly e) (negative numbers only) were initially more difficult than the other (lower scores on the pre-test compared to the other). Although significant progress was made, item d) and e) have lower scores on post-test compared to a-c. There were still students (15–26 %) who did not manage to find the highest number among negative numbers or negative numbers and zero after the lesson.

The analysis of task 3 and 4 (Subtraction pos./neg. difference) suggests improvement on all items except a) c) e) and f). These subtractions (positive difference) are well known to the students, but the encounter with subtractions with negative numbers might have confused some students.

Table 2: Numbers (percentage) of students who answered correctly in task 3 and 4.

Item	a	b*	c	d*	e	f	g*	h*	4
Pre-test	113(97)	15(13)	105(90)	19(16)	115(99)	113(97)	11(9)	17(15)	17(15)
Post-test	96(83)	57(49)	90(78)	62(53)	112(96)	110(95)	51(44)	52(45)	61(52)

N=116 *negative difference

The frequency of correct answers on the subtractions with negative difference are particularly interesting. Item b), d), g) and h) show a similar result. About half of the students could solve these correctly after being taught just one lesson. Before they were taught, in average the frequency of correct answers on these items on the pre-test was slightly more than 10 % (13.6). So, there was a great improvement on the post-test.

Item g) $-2-2=?$ is perhaps the most interesting. This item had the lowest frequency of correct answers before the lesson. Only 9% managed this on the pre-test. 44% answered correctly on the post-test. To be able to solve $-2-2=?$, one must differentiate the minus sign as an operation sign and the sign for negative numbers. Task 9 was designed to test whether the learners could understand the two meanings of the minus sign. On the pre-test 8 students (7%) could tell the difference between the minus sign in the operations $4 - 7 = -3$ and $4 - 3 = 1$ respectively. After the lesson, 42% answered correctly.

Conclusions and discussion.

What has been reported here is not a description of the 'best' lesson design or an answer about to how to teach negative numbers. It is a theoretically and empirically grounded description of some necessary conditions for learning about negative numbers among young students, generated by a group of professionals. Our interpretation of the analysis so far is that, *the simultaneous differentiation of the value of two negative numbers, the differentiation of the function of the minuend and subtrahend, together with differentiating the meaning of the minus sign*, seem to be necessary conditions for learning about the nature of negative numbers. Although learning was improved, still there was a fairly large group of students who seemed not to have learned that which was targeted. So, the 'instructional product' is open for development and improvement.

As was described above, finding the critical aspects is a transactional process comprising the learners, their learning (what they learn), what is targeted, or using Dewey & Bentley's (1949) description: a transaction of the known, the knowing and the known. This was demonstrated in the reported LrS; what was found to be critical emerged as the teachers got deeper understanding of how the learners responded to instruction, what was made possible (and not possible) to learn in the lesson in relation to the targeted 'known'. The object of learning, in terms of what is critical for learning, is constituted in a transactional and continuing process in LrS. The instructional products produced in LrS are hypotheses of what is needed to learn, that can and must be tested and developed to deepen the understanding of teaching and learning. The object of learning and its critical aspects are dynamic and emergent, and this study supports this proposal. In this study, there are most likely things that have been taken for granted or even neglected that might be critical.

Hiebert and Morris (2011) call for a need to accumulate evidence about what works and what does not across different classroom settings (p. 5). Our analysis suggests that, the results for the FS-group on the post-tests reflect the results for the LrS-group after the lesson. Our study supports finding from Kullberg's study (2012); when critical aspects generated in LrS become visible as dimensions of variation in new settings, similar learning outcomes are gained. This further suggests and points to possibilities that the development of effective ways of teaching could be shared among professionals.

References

Altiparmak, K., & Özdoğan, E. (2010). A study on the teaching of the concept of negative numbers. *International Journal of Mathematical Education in Science and Technology, 41*(1), 31–47.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373–397.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19–61

Chichibu, T., & Kihara, T. (2013) How Japanese schools build a professional learning community by lesson study, *International Journal for Lesson and Learning Studies, 2*(1), 12–25.

Dewey, J. & Bentley, A. (1949). *Knowing and the Known. The Later Works, 1949-1952*. Vol 16, pp. 1-280.

Hunter, J., & Back, J. (2011). Facilitating sustainable professional development through lesson study. Mathematics teacher education and development, 13(1), 94–114.

Kullberg, A. (2012). Can findings from learning studies be shared by others? *International Journal for Lesson and Learning Studies, 1*(3), 232–244

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: a theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Lo, M.L., P.P.M. Chik and M.F. Pang (2006), Patterns of Variation in Teaching the Colour of Light to Primary 3 Students. *Instructional Science, 34*(1), 1–19.

Lövström, A. (2015*). Från naturliga tal till hela tal (från N->Z). Vad kan görs skillnad för elevers möjligheter att bli bekanta med de negativa talen?* [In English: From natural numbers to integers (N->Z). What can make a difference to students' possibilities to become familiar with negative numbers? Jönköping: Jönköping University, School of Education and Communication. Research report No. 4.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.

Marton, F., & Pang, M. F. (2003). Beyond "lesson study": Comparing two ways of facilitating the grasp of economic concepts. *Instructional Science, 31*(3), 175–194.

Morris, A. K, & Hiebert, J. (2011). Creating shared instructional products: An alternative approach to improving teaching. *Educational Researcher, 40*(5), 5–14.

Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of 'critical aspects'. *Scandinavian Journal of Educational Research, 60*(3), 323–336.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: The Free Press.

Toshiya, C., & Toshiyuki, K. (2013). How Japanese schools build a professional learning community by lesson study, *International Journal for Lesson and Learning Studies, 2*(1), 12–25.

Watson, A., & Mason, J. (2006). Seeing exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Teaching and Learning, 8*(2), 91–111.